

On the Hybrid Minimum Principle: The Hamiltonian and Adjoint Boundary Conditions

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Abstract—The hybrid minimum principle is presented for the optimal control of deterministic hybrid systems with both autonomous and controlled switchings and jumps where state jumps at the switching instants are permitted to be accompanied by changes in the dimension of the state space. A feature of particular importance is the explicit presentation of the boundary conditions on the Hamiltonians and the adjoint processes before and after switchings and jumps. The numerical benefit of these expressions are demonstrated on a modified version of the multiple autonomous switchings algorithm. The results are illustrated for the hybrid model of an electric vehicle powertrain with a two-speed transmission.

Index Terms—Hybrid systems, minimum principle (MP), nonlinear control systems, optimal control, Pontryagin maximum principle.

I. INTRODUCTION

One of the principal approaches in solving optimal control problems is the Minimum Principle (MP), also called the Maximum Principle in the pioneering work of Pontryagin *et al.* [1], which provides a set of necessary conditions that must be satisfied by all optimal processes. This principle states that along optimal state processes there exist adjoint processes such that their joint governing dynamics possess a Hamiltonian canonical form and that the optimal input process is the point-wise minimizer (or the maximizer depending on the sign convention) of the Hamiltonian function. In other words, the significance of the MP is that it turns the “cost functional minimization” (over the infinite dimensional space of input processes) into a “Hamiltonian function minimization” (over the point-wise value of the input), based upon solutions of a set of two-point boundary value ordinary differential equations (ODEs).

The Minimum Principle, as indicated by the name, is a *principle*, i.e., a not yet completely precise statement that requires technical conditions to be stated as a theorem [2]. For control systems with continuous dynamics these technical conditions are mostly on the regularity requirements (as, e.g., indicated in [3] they are joint conditions on [continuous] state and input processes). For hybrid control systems, however, further technical conditions need to be imposed on interactions of the continuous and discrete subsystems. Various versions of the MP for hybrid systems are available in the control

theory literature [4]–[14]. However, they do not exhaust the full power of the principle. In particular, (i) the presence of both autonomous and controlled switchings, (ii) the possibility of jumps in the state at switching instances, (iii) the possibility of dimension changes in the state space, and (iv) the consideration of switching costs together with running and terminal costs, are characteristics of which only strict subsets have appeared in the literature.

However, there are several engineering systems that exhibit the features (i)–(iv) abovementioned in entirety, or with combinations for which a version of the MP is not immediately available. As an important example, one can refer to the control of electric vehicles equipped with a dual-stage planetary transmission studied in [15]–[18] whose associated hybrid optimal control problem is presented in this article as well. Similar characteristics appear in the extension of this work to stochastic hybrid systems [19] and in hybrid mean field games theory and applications [20], [21] where an agent’s state extended by the mean field terms (associated with active agents) undergoes dimension changes when a group of agents join or leave the population, and where the terminal cost of the leaving agents constitutes a switching cost for the population’s mean field.

The primary objective of this article is the presentation of a general version of the Hybrid Minimum Principle (HMP) for deterministic systems that captures all characteristics (i)–(iv) abovementioned. The regularity assumptions on the continuous dynamics are minimal and imposed primarily to ensure the existence and uniqueness of solutions as well as continuous dependence on initial conditions [22]–[24]. Further generalizations such as the lying of the system’s vector fields in Riemannian spaces [14], nonsmooth assumptions [4], [5], state-dependence of the control value sets [8], and interactions with stochastic subsystems [19], as well as restrictions to certain subclasses, such as those with regional dynamics [25], [26], and with specified families of jumps [27]–[30], become possible through variations and extensions of the framework presented here.

The secondary objective of this work is the explicit expression of the boundary conditions on the Hamiltonians and adjoint processes in contrast to their implicit expressions in the literature in the form of the so-called transversality conditions. This provides a potential to improve the performance of numerical algorithms (e.g., [10], [31]–[40]) that satisfy the Hamiltonian continuity condition implicitly.

The tertiary objective of this note is to illustrate the theoretical results by means of a worked out example of energy minimization for an electric vehicle whose study requires the features (i), (ii), and (iii) abovementioned, due to the addition of a multispeed transmission. More specifically, feature (i) is a necessity since the initiation of gear changing is a controlled switching while the termination of a gear changing process requires the satisfaction of full stop conditions for certain rotary elements, hence is an autonomous switching. Moreover, (ii) and (iii) are essential due to the possession of different mechanical degrees of freedom in each mode and the relationships between the generalized coordinates in each of those modes. Last but not least, the accommodation of (iv) permits the study of hybrid optimal control problems associated with the minimization of the total energy for the acceleration and deceleration of the vehicle with switching costs representing the energy consumption and losses contributed by the electronics operating the locks and brakes inside the transmission mechanism. Further analytic examples can be found in [41]–[43].

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The organization of this article is as follows. The definition of hybrid systems and the associated class of hybrid optimal control problems are, respectively, presented in Sections II and III. The HMP is presented in Section IV in the conventional Hamiltonian canonical form and with boundary conditions in the generalized transversality form. In Section V, an explicit expression for the transversality conditions is presented based upon which a modified version of the multiple autonomous switchings (MAS) algorithm is presented. The representative power of the theoretical framework and the implementation steps of the HMP results are illustrated in Section VI where the energy consumption minimization for an EV equipped with a particular transmission is studied.

II. HYBRID SYSTEMS

A (deterministic) *hybrid system* \mathbb{H} is a septuple

$$\mathbb{H} = \{H, I, \Gamma, A, F, \Xi, \mathcal{M}\} \quad (1)$$

where the symbols in the expression and their governing assumptions are defined as follows.

A0: $H := \prod_{q \in Q} \mathbb{R}^{n_q}$ is called the (*hybrid*) *state space* of the hybrid system \mathbb{H} , where \prod denotes disjoint union, i.e., $\prod_{q \in Q} \mathbb{R}^{n_q} = \bigcup_{q \in Q} \{(q, x) : x \in \mathbb{R}^{n_q}\}$, where $Q = \{1, 2, \dots, |Q|\} \equiv \{q^{(1)}, q^{(2)}, \dots, q^{(|Q|)}\}$, with $|Q| < \infty$, is a finite set of *discrete states (components)*, and

$\{\mathbb{R}^{n_q}\}_{q \in Q}$ is a family of finite dimensional continuous state spaces, where $n_q \leq n < \infty$ for all $q \in Q$.

$I := \Sigma \times U$ is the set of system input values, where

Σ with $|\Sigma| < \infty$ is the set of discrete state transition and continuous state jump events extended with the identity element,

$U = \{U_q\}_{q \in Q}$ is the set of *admissible (continuous) control values*, where each $U_q \subset \mathbb{R}^{m_q}$ is a compact set in \mathbb{R}^{m_q} .

The set of admissible (continuous) control inputs $\mathcal{U}(U) := L_\infty([t_0, T_*], U)$ is defined to be the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*]$, $T_* < \infty$, where the boundedness property necessarily holds, here since admissible inputs take values in the compact set U .

$\Gamma : H \times \Sigma \rightarrow H$ is a time independent (partially defined) *discrete state transition map*.

$\Xi : H \times \Sigma \rightarrow H$ is a time independent (partially defined) *continuous state jump transition map*. For all $\xi \in \Xi$, the functions $\xi_\sigma \equiv \xi(\cdot, \sigma) : \mathbb{R}^{n_q} \rightarrow \mathbb{R}^{n_p}$, $p \in A(q, \sigma)$ are assumed to be continuously differentiable in the continuous state $x \in \mathbb{R}^{n_q}$.

$A : Q \times \Sigma \rightarrow Q$ denotes both a deterministic finite automaton and the automaton's associated transition function on the state space Q and event set Σ , such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q -dependent subset $\{A(q, \sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : Q \times \mathbb{R}^n \times \Sigma \rightarrow H|_Q$. In other words, Γ can only make a discrete state transition in a hybrid state (q, x) if the automaton A can make the corresponding transition in q .

F is an indexed collection of *vector fields* $\{f_q\}_{q \in Q}$ such that for each $q \in Q$ there exist $k_{f_q} \geq 1$ for which $f_q \in C^{k_{f_q}}(\mathbb{R}^{n_q} \times U_q \rightarrow \mathbb{R}^{n_q})$ satisfies a joint uniform Lipschitz condition, i.e., there exists $L_f < \infty$ such that $\|f_q(x_1, u_1) - f_q(x_2, u_2)\| \leq L_f(\|x_1 - x_2\| + \|u_1 - u_2\|)$ for all $x, x_1, x_2 \in \mathbb{R}^{n_q}$, $u, u_1, u_2 \in U_q$.

$\mathcal{M} = \{m_\alpha : \alpha \in Q \times Q\}$ denotes a collection of *switching manifolds* such that, for any ordered pair $\alpha \equiv (\alpha_1, \alpha_2) = (q, r)$, m_α is a smooth, i.e., C^∞ codimension 1 submanifold of \mathbb{R}^{n_q} , described locally by $m_\alpha = \{x : m_\alpha(x) = 0\}$, and possibly with boundary ∂m_α . It is assumed that $m_\alpha \cap m_\beta = \emptyset$, whenever $\alpha_1 = \beta_1$ but $\alpha_2 \neq \beta_2$, for all $\alpha, \beta \in Q \times Q$. \square

Switching manifolds will function in such a way that whenever a trajectory governed by the controlled vector field meets the switching manifold transversally there is an autonomous switching to another controlled vector field or there is a jump transition in the continuous state component, or both. A transversal arrival on a switching manifold $m_{q,r}$,

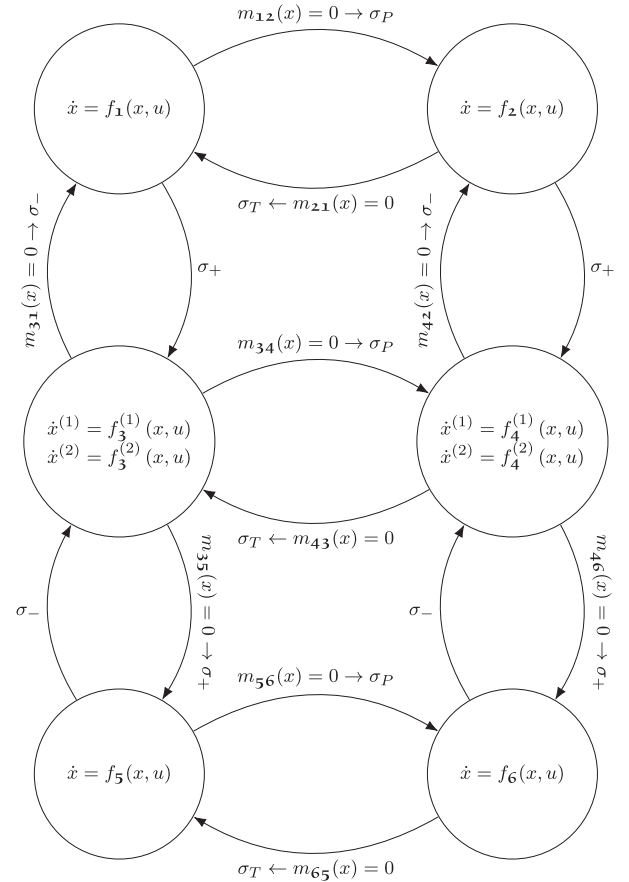


Fig. 1. Hybrid automata diagram for the transmission-equipped electric vehicle in [15] and [17] that serves as an example in Section VI

at state $x_q \in m_{q,r} = \{x \in \mathbb{R}^{n_q} : m_{q,r}(x) = 0\}$ occurs whenever

$$\nabla m_{q,r}(x_q)^\top f_q(x_q, u_q) \neq 0 \quad (2)$$

for $u_q \in U_q$, and $q, r \in Q$. It is assumed that

A1: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0, q_j}(x_0) \neq 0$, for all $q_j \in Q$. \square

A hybrid input process defined over $[t_0, t_f]$, $t_f < \infty$ is denoted by $I_L = (S_L, \mathbf{u})$, where $S_L = ((t_0, \sigma_0), (t_1, \sigma_1), \dots, (t_L, \sigma_L))$, $L < \infty$, is a finite *hybrid sequence of switching events* with $\tau_L := \{t_0, t_1, t_2, \dots, t_L\}$ a strictly increasing sequence of times, $\sigma_i \in \Sigma$, $i \in \{1, 2, \dots, L\}$, with $\sigma_0 = id$, an admissible language of the automata A , and $\mathbf{u} \equiv \{u_{q_0}, u_{q_1}, \dots, u_{q_L}\} \in \mathcal{U}$, with $u_{q_i} \in L_\infty([t_i, t_{i+1}], U_{q_i})$ and admissible input, such that the associated hybrid state trajectory satisfies.

A. Continuous State Dynamics

The continuous component of the hybrid state $\mathbf{x} = \{x_{q_0}(\cdot), x_{q_1}(\cdot), \dots, x_{q_L}(\cdot)\}$ is a piecewise continuous function, which is almost everywhere differentiable and on each time segment specified by τ_L satisfies the dynamical equation

$$\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \quad a.e. t \in [t_i, t_{i+1}) \quad (3)$$

with the initial conditions

$$x_{q_0}(t_0) = x_0 \quad (4)$$

$$x_{q_i}(t_i) = \xi_{q_{i-1}q_i}(x_{q_{i-1}}(t_{i-1}^-)) := \xi_{q_{i-1}q_i} \left(\lim_{t \uparrow t_i} x_{q_{i-1}}(t) \right). \quad (5)$$

B. Autonomous Discrete Transition Dynamics

An autonomous (uncontrolled) discrete state transition from q_{i-1} to q_i together with a continuous state jump $\xi_{q_{i-1}q_i}$ occurs at the *autonomous switching time* t_i if $x_{q_{i-1}}(t_i^-) := \lim_{t \uparrow t_i} x_{q_{i-1}}(t)$ satisfies a switching manifold condition of the form

$$m_{q_{i-1}q_i}(x_{q_{i-1}}(t_i^-), t_i) = 0 \quad (6)$$

for $q_i \in Q$, where $m_{q_{i-1}q_i}(x) = 0$ defines a (q_{i-1}, q_i) switching manifold and it is not the case that either (i) $x(t_i^-) \in \partial m_{q_{i-1}q_i}$ or (ii) $f_{q_{i-1}}(x(t_i^-), u(t_i^-)) \perp \nabla m_{q_{i-1}q_i}(x(t_i^-))$, i.e., t_i is not a manifold termination instant (see [44]). With the Assumptions A0 and A1 in force, such a transition is well defined and labels the event $\sigma_{q_{i-1}q_i} \in \Sigma$, which corresponds to the hybrid state transition

$$\begin{aligned} h(t_i) &\equiv (q_i, x_{q_i}(t_i)) \\ &= (\Gamma(q_{i-1}, x_{q_{i-1}}(t_i^-), \sigma_i), \xi_{q_{i-1}q_i}(x_{q_{i-1}}(t_i^-))). \end{aligned} \quad (7)$$

C. Controlled Discrete Transition Dynamics

A controlled discrete state transition together with a controlled continuous state jump $\xi_{q_{i-1}q_i}$ occurs at the *controlled discrete event time* t_i if t_i is not an autonomous discrete event time and if there exists a controlled discrete input event $\sigma_{q_{i-1}q_i} \in \Sigma$ for which

$$\begin{aligned} h(t_i) &\equiv (q_i, x_{q_i}(t_i)) \\ &= (\Gamma(q_{i-1}, x_{q_{i-1}}(t_i^-), \sigma_i), \xi_{q_{i-1}q_i}(x_{q_{i-1}}(t_i^-))) \end{aligned} \quad (8)$$

with $(t_i, \sigma_{q_{i-1}q_i}) \in S_L$ and $q_i \in A(q_{i-1})$. \square

A2: For a specified sequence of discrete states $\{q_i\}_{i=0}^L$, the class of admissible input-state trajectories is nonempty. \square

Theorem 2.1: [44] A hybrid system \mathbb{H} with an initial hybrid state (q_0, x_0) satisfying Assumptions A0–A2 possesses a unique hybrid input-state trajectory on $[t_0, T_{**})$, where T_{**} is the least of

- i) $T_* \leq \infty$, where $[t_0, T_*)$ is the temporal domain of the definition of the hybrid system;
- ii) a manifold termination instant T_* of the trajectory $h(t) = h(t, (q_0, x_0), (S_L, u))$, $t \geq t_0$, at which either $x(T_*^-) \in \partial m_{q(T_*^-)q(T_*)}$ or $f_{q(T_*^-)}(x(T_*^-), u(T_*^-)) \perp \nabla m_{q(T_*^-)q(T_*)}(x(T_*^-))$. \square

We note that Zeno times, i.e., accumulation points of discrete transition times, are ruled out by A2.

III. HYBRID OPTIMAL CONTROL PROBLEMS

A3: Let $\{l_q\}_{q \in Q}, l_q \in C^{n_l}(\mathbb{R}^n \times U \rightarrow \mathbb{R}_+)$, $n_l \geq 1$, be a family of running cost functions; $\{c_\sigma\}_{\sigma \in \Sigma}, c_\sigma \in C^{n_c}(\mathbb{R}^n \times \Sigma \rightarrow \mathbb{R}_+)$, $n_c \geq 1$, be a family of switching cost functions; and $\{g_q\}_{q \in Q}, g_q \in C^{n_g}(\mathbb{R}^n \rightarrow \mathbb{R}_+)$, $n_g \geq 1$, be a family of terminal cost functions satisfying the following assumptions.

- i) There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $|l_q(x, u)| \leq K_l(1 + \|x\|^{\gamma_l})$ and $|l_q(x_1, u_1) - l_q(x_2, u_2)| \leq K_l(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $q \in Q, x \in \mathbb{R}^n, u \in U_q$.
- ii) There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c_\sigma(x)| \leq K_c(1 + \|x\|^{\gamma_c})$, $\sigma \in \Sigma, \sigma_1 = q, x \in \mathbb{R}^n$.
- iii) There exists $K_g < \infty$ and $1 \leq \gamma_g < \infty$ such that $|g_q(x)| \leq K_g(1 + \|x\|^{\gamma_g})$, $q \in Q, x \in \mathbb{R}^n$. \square

Consider the initial time t_0 , final time $t_f < \infty$, and initial hybrid state $h_0 = (q_0, x_0)$. With the number of switchings L held fixed, the set of all hybrid input trajectories with exactly L switchings is denoted by I_L . Let $I_L \in I_L$ be a hybrid input trajectory that by Theorem 2.1 results in a unique hybrid state process. Then, hybrid performance functions for

the corresponding hybrid input-state trajectory are defined as follows:

$$\begin{aligned} J(t_0, t_f, h_0, L; I_L) &:= \sum_{i=0}^L \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s), s) ds \\ &+ \sum_{j=1}^L c_{q_{j-1}q_j}(t_j, x_{q_{j-1}}(t_j^-)) + g(x_{q_L}(t_f)). \end{aligned} \quad (9)$$

IV. HYBRID MINIMUM PRINCIPLE

Theorem 4.1 ([45]): Consider the hybrid system \mathbb{H} subject to Assumptions A0–A3, and the HOCBP with the hybrid performance function (9). Define the family of system Hamiltonians by

$$H_q(x_q, \lambda_q, u_q, t) = l_q(x_q, u_q, t) + \lambda_q^\top f_q(x_q, u_q, t) \quad (10)$$

$x_q, \lambda_q \in \mathbb{R}^{n_q}, u_q \in U_q, q \in Q$, and let $\{q_i\}_{i=0}^L$ be a specified sequence of discrete states with its associated set of switchings. Then, for an optimal input u° and along the corresponding optimal trajectory x° , there exists an adjoint process λ° such that

$$H_q(x_q^\circ, \lambda_q^\circ, u_q^\circ, t) \leq H_q(x_q^\circ, \lambda_q^\circ, v, t) \quad (11)$$

for all $v \in U_q$, where (x°, λ°) satisfy

$$\dot{x}_q^\circ = \frac{\partial H_q}{\partial \lambda_q}(x_q^\circ, \lambda_q^\circ, u_q^\circ, t) \equiv f_q(x_q^\circ, u_q^\circ, t) \quad (12)$$

$$\begin{aligned} \dot{\lambda}_q^\circ &= -\frac{\partial H_q}{\partial x_q}(x_q^\circ, \lambda_q^\circ, u_q^\circ, t) \\ &\equiv -\frac{\partial l_q}{\partial x}(x_q^\circ, u_q^\circ, t) - \frac{\partial f_q}{\partial x}(x_q^\circ, u_q^\circ, t)^\top \lambda_q^\circ \end{aligned} \quad (13)$$

almost everywhere $t \in [t_0, t_f]$, subject to

$$x_{q_0}^\circ(t_0) = x_0 \quad (14)$$

$$x_{q_j}^\circ(t_j) = \xi_{q_{j-1}q_j}(x_{q_{j-1}}^\circ(t_{j-1}^-)) \quad (15)$$

$$\lambda_{q_L}^\circ(t_f) = \nabla g(x_{q_L}^\circ(t_f)) \quad (16)$$

$$\begin{aligned} \lambda_{q_{j-1}}^\circ(t_{j-1}^-) &\equiv \lambda_{q_{j-1}}^\circ(t_j) = \nabla \xi_{q_{j-1}q_j}^\top \lambda_{q_j}^\circ(t_j^+) \\ &+ \nabla c_{q_{j-1}q_j} + p_j \nabla m_{q_{j-1}q_j} \end{aligned} \quad (17)$$

where $p_j \in \mathbb{R}$ when t_j indicates the time of an autonomous switching, subject to the switching manifold condition $m_{q_{j-1}q_j}(x_{q_{j-1}}^\circ(t_{j-1}^-)) = 0$, and $p_j = 0$ when t_j indicates the time of a controlled switching. Moreover, the Hamiltonian satisfies

$$\begin{aligned} H_{q_{j-1}}(t_{j-1}^-) &\equiv H_{q_{j-1}}(x_{q_{j-1}}^\circ, \lambda_{q_{j-1}}^\circ, u_{q_{j-1}}^\circ, t_{j-1}^-) \\ &= H_{q_j}(x_{q_j}^\circ, \lambda_{q_j}^\circ, u_{q_j}^\circ, t_j^+) - \frac{\partial c_{q_{j-1}q_j}}{\partial t} - p_j \frac{\partial m_{q_{j-1}q_j}}{\partial t} \\ &\equiv H_{q_j}(t_j^+) - \frac{\partial c_{q_{j-1}q_j}}{\partial t} - p_j \frac{\partial m_{q_{j-1}q_j}}{\partial t} \end{aligned} \quad (18)$$

at both autonomous and controlled switching instants t_j . \square

V. EXPLICIT EXPRESSIONS AND THE ASSOCIATED HMP-MAS ALGORITHM

Let $N := \sum_{i=0}^L n_{q_i}$. We remark that in the forward-backward ODEs (12) and (13), there is a total of $2N$ state-type components for which N initial conditions are provided by (14) and (15) and N terminal conditions are provided by (16) and (17). There are also L unknown (not *a priori* fixed) values for t_j for which there are L algebraic conditions provided by (18). Moreover, for every case of an autonomous switching,

there is an unknown scalar p_j whose values can be uniquely determined from the simultaneous solution of the set of equations together with the set of constraints imposed by (6) whose count is the same as the number of the unknown values of p_j 's. Hence, the set of necessary conditions in the HMP is complete in the sense that the number of unknowns match the number of conditions they must satisfy. However, the implicit determination of p_j 's from the holistic set of differential-algebraic equations might not translate well in some numerical solution methodologies.

A. Explicit Expressions for the Hamiltonians and the Adjoints Boundary Conditions

As obtained in the authors' proof of the HMP (see, e.g., [45]), the scalar p_j in the adjoint boundary conditions (17) and the Hamiltonian boundary conditions can be alternatively determined from

$$p_j := \frac{l_{q_j, \xi}^{q_j-1} + \lambda_{q_j}^\top f_{q_j, \xi}^{\xi, q_j-1} - \partial c_{t, x}^{1, f_{q_j-1}}}{\frac{\partial m_{q_j-1, q_j}}{\partial t} + \left[\frac{\partial m_{q_j-1, q_j}}{\partial x_{q_j-1}} \right]^\top f_{q_j-1}} \quad (19)$$

for every autonomous switching instant t_j , where

$$l_{q_j, \xi}^{q_j-1} := l_{q_j}(\xi(x), u(\xi(x), \lambda_{q_j})) - l_{q_j-1}(x, u(x, \lambda_{q_j-1})) \quad (20)$$

$$f_{q_j, \xi}^{\xi, q_j-1} := f_{q_j}(\xi(x), u(\xi(x), \lambda_{q_j})) - \nabla \xi f_{q_j-1}(x, u(x, \lambda_{q_j-1})) \quad (21)$$

$$\partial c_{t, x}^{1, f_{q_j-1}} := \frac{\partial}{\partial t} c_{q_j-1, q_j}(x) - \nabla c_{q_j-1, q_j}^\top f_{q_j-1}(x, u(x, \lambda_{q_j-1})) \quad (22)$$

and where

$$u(x, \lambda) := \operatorname{argmin}_{u \in U_{q(t)}} H_{q(t)}(x, \lambda, u, t) \quad (23)$$

is used in (20)–(22) for the ease of notation.

B. HMP–MAS Algorithm

The HMP-based Multiple Autonomous Switchings algorithm in [10] has been originally developed for a class of hybrid systems with the feature (i) mentioned in Section I, but not covering the features (ii)–(iv). In this section, we present a generalization of the HMP–MAS algorithm that covers a general class of hybrid optimal control problems with the features (i)–(iv), while also making a modification based upon the explicit expression (19) presented in the previous section.

Let $\{q_i\}_{i=0}^L$ be the given discrete state sequence and let $\{(\hat{t}_i, \hat{x}_{q_{i-1}}(\hat{t}_i-))\}_{i=1}^L$ be a nominal set of feasible (but not necessarily optimal) switching times and states. By feasibility we mean that (a) for every autonomous switching pair, $(\hat{t}_i, \hat{x}_{q_{i-1}}(\hat{t}_i-))$ the corresponding switching manifold condition $m_{q_{i-1}q_i}((\hat{t}_i, \hat{x}_{q_{i-1}}(\hat{t}_i-))) = 0$ is satisfied, and (b) for every switching time \hat{t}_{i+1} the associated preswitching state $\hat{x}_{q_i}(\hat{t}_{i+1}-)$ is reachable from the previous point, i.e., there exist some nominal $\hat{u}_s, s \in [\hat{t}_i, \hat{t}_{i+1})$ such that

$$\hat{x}_{q_i}(\hat{t}_{i+1}-) = \xi_{q_{i-1}q_i}(\hat{x}_{q_{i-1}}(\hat{t}_i-) + \int_{\hat{t}_i}^{\hat{t}_{i+1}} f_{q_i}(\hat{x}_s, \hat{u}_s) ds. \quad (24)$$

It is, however, not essential in the initiation step to generate such $\hat{u}_s, s \in [\hat{t}_i, \hat{t}_{i+1})$ and only the reachability of $\hat{x}_{q_i}(\hat{t}_{i+1}-)$ from $\xi_{q_{i-1}q_i}(\hat{x}_{q_{i-1}}(\hat{t}_i-))$ is a sufficient information.

0) Algorithm Initiation: Fix the termination tolerance $\epsilon_f > 0$ sufficiently small, a monotonically nondecreasing sequence of step sizes $\{r_k\}$ with $r_k < 1$, and set the iteration counter $k = 0$. Set $\hat{t}_i^k = \hat{t}_i$ and $y_i^k = \hat{x}_{q_{i-1}}(\hat{t}_i-)$. We also use the notations $t_0^k = t_0$, $\xi_{q_{-1}q_0}(y_0) = x_0$ and $t_{L+1}^k = t_f$.

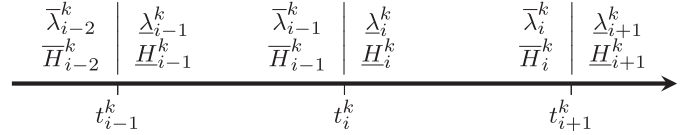


Fig. 2. Illustration of the notations used for the adjoints and the Hamiltonians at iteration k within the intervals $[t_i^k, t_{i+1}^k]$.

1) Multiple Two-Point Boundary Value Problems (TP–BVP): Solve the set of TPBVP associated with each q_{i-1} over $[t_{i-1}^k, t_i^k]$, $i \in \{1, 2, \dots, L\}$, with fixed initial and terminal states $\xi_{q_{i-2}q_{i-1}}(y_{i-1}^k)$ and y_i^k , where these TPBVPs are decoupled in the sense that their adjoint processes and Hamiltonians are not related to each other. Obtain the initial $\lambda_{q_{i-1}}^k(t_{i-1}^k)$, $H_{q_{i-1}}^k(t_{i-1}^k)$ and terminal values $\lambda_{q_{i-1}}^k(t_i^k)$, $H_{q_{i-1}}^k(t_i^k)$ for the adjoints and the Hamiltonians of each of the decoupled TPBVPs that, for convenience of notation, are denoted by (see Fig. 2)

$$\underline{\lambda}_{i-1}^k := \lambda_{q_{i-1}}^k(t_{i-1}^k) \quad (25)$$

$$\bar{\lambda}_{i-1}^k := \lambda_{q_{i-1}}^k(t_i^k) \quad (26)$$

$$\underline{H}_{i-1}^k := H_{q_{i-1}}^k(t_{i-1}^k, \xi(y_{i-1}^k), u(\xi(y_{i-1}^k), \lambda_{q_{i-1}}^k(t_{i-1}^k))) \quad (27)$$

$$\bar{H}_{i-1}^k := H_{q_{i-1}}^k(t_i^k, y_i^k, u(y_i^k, \lambda_{q_{i-1}}^k(t_i^k))). \quad (28)$$

This step requires access to a classical (nonhybrid) but adjoint-based optimal control solver such as shooting-based methods.

2) Updating Procedure: Obtain new switching pairs from

$$t_i^{k+1} = t_i^k - r_k \left(\bar{H}_{i-1}^k + p_i^k \frac{\partial m_i^k}{\partial t} - \underline{H}_i^k \right) - r_k m_i^k \frac{\partial m_i^k}{\partial t} \quad (29)$$

$$y_i^{k+1} = y_i^k - r_k \left(\nabla \xi_{i,k}^\top \underline{\lambda}_i^k + \nabla c_i^k + p_i^k \nabla m_i^k - \bar{\lambda}_{i-1}^k \right) - r_k m_i^k \nabla m_i^k \quad (30)$$

where

$$p_i^k = \frac{\underline{H}_i^k - \bar{H}_{i-1}^k + f_{i-1,k}^\top (\bar{\lambda}_{i-1}^k - \nabla \xi_{i,k}^\top \underline{\lambda}_i^k - \nabla c_i^k) - \partial c_i / \partial t}{\partial m_i^k / \partial t + f_{i-1,k}^\top \nabla m_i^k} \quad (31)$$

$$f_{i-1,k} := f_{q_{i-1}}(t_i^k, y_i^k, u(y_i^k, \bar{\lambda}_{i-1}^k)) \quad (32)$$

$$\xi_{i,k} := \xi_{q_{i-1}q_i}(y_i^k) \quad (33)$$

$$c_i^k := c_{q_{i-1}q_i}(y_i^k) \quad (34)$$

$$m_i^k := m_{q_{i-1}q_i}(y_i^k). \quad (35)$$

The updates (29) and (30) are descent directions for the cost

$$\mu_k = \sum_{i=1}^L \left\| \left[\bar{H}_{i-1}^k + p_i^k \frac{\partial m_i^k}{\partial t} - \underline{H}_i^k \right] \right\|^2 + |m_i^k|^2 \quad (36)$$

whose minimizer is a set of switching times and pre-switching states (t_i^*, y_i^*) that satisfy the HMP conditions, but at intermediate steps the switching manifold conditions might be violated. See [32] for a geodesic gradient flow algorithm that overcomes this disadvantage.

VI. ELECTRIC VEHICLE WITH TRANSMISSION

For the illustration of the representative power of the presented HMP framework, we study an electric vehicle equipped with a dual planetary transmission presented in [15] whose hybrid systems formulation is developed in [17]. While a detailed derivation is presented in [17], the emphasize in this article is on the control theoretic aspects of the associated hybrid optimal control problem.

A. Hybrid Systems Presentation of the Powertrain

The automata diagram of powertrain of an EV equipped with a dual planetary transmission is illustrated in Fig. 1. For this system, the discrete state set $Q = \{1, 2, 3, 4, 5, 6\}$ where the name of modes are denoted by i instead of q_i to avoid ambiguity when referring to q_i as the value of the discrete mode in the time interval $[t_i, t_{i+1})$. The discrete states 1, 2, 5, 6 correspond to fixed gear ratios, whereas 3, 4 represent the system dynamics in transition between gears. The modes 1, 3, 5 correspond to the operation of the electric motor in low speeds where the motor torque is limited by a maximum torque constraint and the modes 2, 4, 6 correspond to the operation of the electric motor in high speeds where the motor torque is limited by a maximum power constraint.

During the transition between the gears, the powertrain possesses one more degree of freedom than fixed gear ratio modes, and hence the simplest hybrid state space (with the car velocity as its state in fixed gear modes and with two independent angular velocities as the state in the transition modes) gives the hybrid state space $H = \{(1, \mathbb{R}), (2, \mathbb{R}), (3, \mathbb{R}^2), (4, \mathbb{R}^2), (5, \mathbb{R}), (6, \mathbb{R})\}$.

The discrete input set $\Sigma = \{\sigma_+, \sigma_-, \sigma_P, \sigma_T\}$ where σ_+ leads the system toward a higher speed gear, σ_- directs it toward a lower speed gear, σ_P enforces the maximum power constrained operation, and σ_T enforces the maximum torque constrained operation.

In the discrete states corresponding to fixed gear ratios, the normalized motor torque is the only input and hence, $U_1 = U_2 = U_5 = U_6 = [-1, 1] \subset \mathbb{R}$, where negative values correspond to the regeneration mode of the electric motor. In the discrete states corresponding to transition phases, the normalized forces of the two brakes operating the transmission are inputs, in addition to the motor torque and hence, $U_3 = U_4 = [-1, 1] \times [0, 1] \times [0, 1] \subset \mathbb{R}^3$ where, obviously, the last two components of the input corresponding to the brake forces do not change signs.

The discrete state transition map Γ and the finite automaton A are illustrated in Fig. 1. As can be observed in this figure, Γ can only make a discrete state transition in a hybrid state (q, x) if the automaton A can make the corresponding transition in q .

The elements in the set of vector fields F are given as follows:

$$f_1(x, u) = -A_1x^2 + B_1u - C_1x - D_1 \quad (37)$$

$$f_2(x, u) = -A_1x^2 + B_1\frac{u}{x} - C_1x - D_1 \quad (38)$$

$$\begin{aligned} f_3^{(1)}(x, u) &= -A_{ss}x^{(1)} + A_{sr}x^{(2)} - A_{sa}(x^{(1)} + R_2x^{(2)})^2 \\ &\quad + B_{smt}u^{(1)} + B_{sst}u^{(2)} - B_{srt}u^{(3)} - D_{st} \\ f_3^{(2)}(x, u) &= A_{rs}x^{(1)} - A_{rr}x^{(2)} - A_{ra}(x^{(1)} + R_2x^{(2)})^2 \\ &\quad + B_{rmt}u^{(1)} - B_{rst}u^{(2)} - B_{rrt}u^{(3)} - D_{rl} \end{aligned} \quad (39)$$

$$\begin{aligned} f_4^{(1)}(x, u) &= -A_{ss}x^{(1)} + A_{sr}x^{(2)} - A_{sa}(x^{(1)} + R_2x^{(2)})^2 \\ &\quad + B_{sm}\frac{u^{(1)}}{x^{(1)} + R_1x^{(2)}} + B_{ss}u^{(2)} - B_{sr}u^{(3)} - D_{st} \\ f_4^{(2)}(x, u) &= A_{rs}x^{(1)} - A_{rr}x^{(2)} - A_{ra}(x^{(1)} + R_2x^{(2)})^2 \\ &\quad + B_{rm}\frac{u^{(1)}}{x^{(1)} + R_1x^{(2)}} - B_{rs}u^{(2)} + B_{rr}u^{(3)} - D_{rl} \end{aligned} \quad (40)$$

$$f_5(x, u) = -A_2x^2 + B_2u - C_2x - D_2 \quad (41)$$

$$f_6(x, u) = -A_2x^2 + B_2\frac{u}{x} - C_2x - D_2 \quad (42)$$

where the coefficients A_\bullet , B_\bullet , C_\bullet , D_\bullet , and R_\bullet in the above-mentioned equations and r_\bullet and k_\bullet in the following equations are model parameters whose numerical values can be found in [17].

The set of jump transition maps Ξ is identified by

$$\xi_{12} = \xi_{21} = \text{id}_{\mathbb{R}} : x \rightarrow x \quad (43)$$

$$\xi_{13} = \xi_{24} : x \rightarrow \begin{bmatrix} r_1x \\ 0 \end{bmatrix} \quad (44)$$

$$\xi_{31} = \xi_{42} : \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \rightarrow \frac{x^{(1)}}{r_1} \quad (45)$$

$$\xi_{34} = \xi_{43} = \text{id}_{\mathbb{R}^2} : \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \rightarrow \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \quad (46)$$

$$\xi_{35} = \xi_{46} : \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \rightarrow r_2x^{(2)} \quad (47)$$

$$\xi_{53} = \xi_{64} : x \rightarrow \begin{bmatrix} 0 \\ \frac{x}{r_2} \end{bmatrix} \quad (48)$$

$$\xi_{56} = \xi_{65} = \text{id}_{\mathbb{R}} : x \rightarrow x. \quad (49)$$

While initiations of gear changing can be made freely (and therefore switchings to 3, 4 are controlled), the transitions back to a fixed gear mode require the full stop for one of the degrees of freedom. Moreover, switchings between torque-constrained and power-constrained modes occur whenever the motor speed reaches a certain value. The set of switching manifolds \mathcal{M} for the autonomous switchings are given by

$$m_{12} = m_{21} \equiv \{x \in \mathbb{R} : x - k_1 = 0\} \cup \{x \in \mathbb{R} : x + k_1 = 0\} \quad (50)$$

$$m_{31} = m_{42} \equiv \{x \in \mathbb{R}^2 : [0 \ 1] x = 0\} \quad (51)$$

$$\begin{aligned} m_{34} = m_{43} &\equiv \{x \in \mathbb{R}^2 : [1 \ R_1] x - k_2 = 0\} \\ &\cup \{x \in \mathbb{R}^2 : [1 \ R_1] x + k_2 = 0\} \end{aligned} \quad (52)$$

$$m_{35} = m_{46} \equiv \{x \in \mathbb{R}^2 : [1 \ 0] x = 0\} \quad (53)$$

$$m_{56} = m_{65} \equiv \{x \in \mathbb{R} : x - k_3 = 0\} \cup \{x \in \mathbb{R} : x + k_3 = 0\}. \quad (54)$$

B. Association of Costs

Depending on the goal, one can associate numerous optimal control problems for the powertrain. For time optimal tasks, the running costs shall be taken to be $l_q(x, u) = 1$ for all $q \in Q \equiv \{1, 2, 3, 4, 5, 6\}$, so that once integrated, their sum gives the total spent time (see [17] and [18] for examples of this class). For the minimization of energy consumption, the running costs shall be taken to be the power consumption rates that are determined from the motor efficiency map (see [17] for the derivation and more discussion). The resulting expressions for l_q 's are presented as follows:

$$l_1(x, u) = a_1u^2 + b_1xu + c_1u + d_1x \quad (55)$$

$$l_2(x, u) = a_1\frac{u^2}{x^2} + b_1u + c_1\frac{u}{x} + d_1x \quad (56)$$

$$\begin{aligned} l_3(x, u) &= a_{tr}(u^{(1)})^2 + b_{tr}u^{(1)}(x^{(1)} + R_1x^{(2)}) \\ &\quad + c_{tr}u^{(1)} + d_{tr}(x^{(1)} + R_1x^{(2)}) \end{aligned} \quad (57)$$

$$l_4(x, u) = a_{tr}\frac{(u^{(1)})^2}{(x^{(1)} + R_1x^{(2)})^2} + b_{tr}u^{(1)}$$

$$+ c_{tr} \frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} + d_{tr} (x^{(1)} + R_1 x^{(2)}) \quad (58)$$

$$l_5(x, u) = a_2 u^2 + b_2 u + c_2 x + d_2 x \quad (59)$$

$$l_6(x, u) = a_2 \frac{u^2}{x^2} + b_2 u + c_2 \frac{u}{x} + d_2 x. \quad (60)$$

For time optimal goals, one can associate (potentially state-dependent) switching costs by considering unaccounted delays in switching. For energy optimal goals, switching costs represent the energy consumption of the mechanism performing the engagement and release of the lock holding the stationary parts fixed. While the HMP framework permits a wide range of nonlinear costs that depend on the rotational speed of transmission elements (gears), we consider a quadratic model fit, i.e.,

$$c_{13}(x) = c_{24}(x) = c_{53}(x) = c_{64}(x) = \eta_0 + \eta_1 x + \eta_2 x^2 \quad (61)$$

$$c_{31}(x) = c_{42}(x) = \eta'_0 + \eta'_1 x^{(1)} + \eta'_2 (x^{(1)})^2 \quad (62)$$

$$c_{35}(x) = c_{46}(x) = \eta''_0 + \eta''_1 x^{(2)} + \eta''_2 (x^{(2)})^2. \quad (63)$$

Notice that the switching cost in (62) contains only the first component of the state $x^{(1)}$ because the second component $x^{(2)}$ corresponds to the speed of the common ring gear in the transmission vanishes as it needs to come to a full stop for the switching to occur. Similarly, (63) contains only the second component of the state $x^{(2)}$ because the common sun gear needs to come to a full stop at these switching instances. For the numerical simulations in this section, we assume a quadratic representation for the terminal cost

$$g(x(t_f)) = \alpha_0 + \alpha_1 x(t_f) + \alpha_2 x(t_f)^2 \quad (64)$$

while any nonlinear representation satisfying A3 is also permitted.

C. HMP Formulation

We consider the energy consumption minimization for acceleration in the first gear from the stationary state, i.e., $h_0 \equiv (q, x)(t_0) = (\mathbf{1}, 0)$ for a period of 2 s, i.e., $[t_0, t_f] = [0, 2]$. We would like to perform a gear change to the second gear ratio, which corresponds to 6 since we know the vehicle's speed exceeds 12 m/s = 43.2 km/h that results in the operation of the electric motor in the power constrained region. Based upon the hybrid automata in Fig. 1, potential sequences of length $L = 4$ for the discrete state are **1356**, **1246**, **1346**. Thus, the HMP needs to be solved three times (one for each sequence) in order to determine the global optimal sequence together with the associated inputs. It turns out that (see [17]) **1246** is the optimal sequence and hence, we present the analysis only for this sequence, while the results for other sequences follow the same derivation.

1) Formation of the Hamiltonians: The family of system Hamiltonians are formed as follows:

$$H_1(x, \lambda, u) = \lambda (-A_1 x^2 + B_1 u - C_1 x - D_1) + a_1 u^2 + b_1 x u + c_1 u + d_1 x \quad (65)$$

$$H_2(x, \lambda, u) = \lambda \left(-A_1 x^2 + B_1 \frac{u}{x} - C_1 x - D_1 \right) + a_1 \frac{u^2}{x^2} + b_1 u + c_1 \frac{u}{x} + d_1 x \quad (66)$$

$$H_4(x, \lambda, u) = \left(-A_{ss} x^{(1)} + A_{sr} x^{(2)} - A_{sa} (x^{(1)} + R_2 x^{(2)})^2 + B_{sm} \frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} + B_{ss} u^{(2)} - B_{sr} u^{(3)} - D_{st} \right) \lambda^{(1)} + \left(A_{rs} x^{(1)} - A_{rr} x^{(2)} - A_{ra} (x^{(1)} + R_2 x^{(2)})^2 \right. \\ \left. + B_{rm} \frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} - B_{rs} u^{(2)} + B_{rr} u^{(3)} - D_{rl} \right) \lambda^{(2)} \\ + a_{tr} \frac{(u^{(1)})^2}{(x^{(1)} + R_1 x^{(2)})^2} + b_{tr} u^{(1)} + c_{tr} \frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} + d_{tr} (x^{(1)} + R_1 x^{(2)}) \quad (67)$$

$$H_6(x, \lambda, u) = \lambda \left(-A_2 x^2 + B_2 \frac{u}{x} - C_2 x - D_2 \right) + a_2 \frac{u^2}{x^2} + b_2 u + c_2 \frac{u}{x} + d_2 x. \quad (68)$$

2) Hamiltonian Minimization:: The Hamiltonian minimization condition (11) for the Hamiltonians (65)–(68) determines the optimal inputs as follows:

$$u_1(t) = \text{sat}_{[-1,1]} \left(\frac{-(b_1 x(t) + c_1 + B_1 \lambda(t))}{2a_1} \right) \quad (69)$$

$$u_2(t) = \text{sat}_{[-1,1]} \left(\frac{-x(t)(b_1 x(t) + c_1 + B_1 \lambda(t))}{2a_1} \right) \quad (70)$$

$$u_4^{(1)}(t) = \text{sat}_{[-1,1]} \left(\frac{-X_M(t) [b_{tr} X_M(t) + c_{tr} + B_{sm} \lambda^{(1)}(t) + B_{rm} \lambda^{(2)}(t)]}{2a_{tr}} \right) \\ u_4^{(2)}(t) = \begin{cases} -1 & \text{if } B_{ss} \lambda^{(1)}(t) - B_{rs} \lambda^{(2)}(t) \geq 0 \\ 0 & \text{if } B_{ss} \lambda^{(1)}(t) - B_{rs} \lambda^{(2)}(t) < 0 \end{cases} \quad (71)$$

$$u_4^{(3)}(t) = \begin{cases} -1 & \text{if } B_{rr} \lambda^{(2)}(t) - B_{sr} \lambda^{(1)}(t) \geq 0 \\ 0 & \text{if } B_{rr} \lambda^{(2)}(t) - B_{sr} \lambda^{(1)}(t) < 0 \end{cases}$$

$$u_6(t) = \text{sat}_{[-1,1]} \left(\frac{-x(t)(b_2 x(t) + c_2 + B_2 \lambda(t))}{2a_2} \right) \quad (72)$$

where $x(t) \equiv x_{q_i}^o(t)$ and $X_M(t) := (x^{(1)}(t) + R_1 x^{(2)}(t))$ are employed to shorten the notation.

3) Adjoint Dynamics: The dynamics of the (backward) adjoint processes are derived from (13) as follows:

$$\dot{\lambda}_1 = -b_1 u_1^o(t) - d_1 + \lambda_1(t) (2A_1 x_1(t) + C_1) \quad (73)$$

$$\dot{\lambda}_2 = \frac{2a_1 (u_2^o(t))^2}{(x_2(t))^3} + \frac{c_1 u_2^o(t)}{(x_2(t))^2} - d_1 + \lambda_2(t) \left(2A_1 x_2(t) + B_1 \frac{u_2^o(t)}{(x_2(t))^2} + C_1 \right) \quad (74)$$

$$\dot{\lambda}_4^{(1)} = \frac{2a_{tr} (u_4^{o(1)}(t))^2}{(x^{(1)} + R_1 x^{(2)})^3} + \frac{c_{tr} u_4^{o(1)}(t)}{(x^{(1)} + R_1 x^{(2)})^2} - d_{tr} + \lambda_4^{(1)} \left(A_{ss} + 2A_{sa} (x^{(1)} + R_2 x^{(2)}) + \frac{B_{sm} u_4^{o(1)}}{(x^{(1)} + R_1 x^{(2)})^2} \right) + \lambda_4^{(2)} \left(-A_{rs} + 2A_{ra} (x^{(1)} + R_2 x^{(2)}) + \frac{B_{rm} u_4^{o(1)}}{(x^{(1)} + R_1 x^{(2)})^2} \right) \quad (75)$$

$$\dot{\lambda}_4^{(2)} = \left(\frac{2R_1 a_{tr} (u_4^{o(1)}(t))^2}{(x^{(1)} + R_1 x^{(2)})^3} + \frac{R_1 c_{tr} u_4^{o(1)}(t)}{(x^{(1)} + R_1 x^{(2)})^2} - R_1 d_{tr} \right) + \lambda_4^{(1)} \left(-A_{sr} + 2R_2 A_{sa} (x^{(1)} + R_2 x^{(2)}) + \frac{R_1 B_{sm} u_4^{o(1)}}{(x^{(1)} + R_1 x^{(2)})^2} \right)$$

$$+\lambda_4^{(2)} \left(A_{rr} + 2R_2 A_{ra} (x^{(1)} + R_2 x^{(2)}) + \frac{R_1 B_{rm} u_4^{o(1)}}{(x^{(1)} + R_1 x^{(2)})^2} \right) \quad (76)$$

$$\dot{\lambda}_6 = \frac{2a_2 (u_6^o(t))^2}{(x_6(t))^3} + \frac{c_2 u_6^o(t)}{(x_6(t))^2} - d_2 + \lambda_6(t) \left(2A_2 x_6(t) + B_2 \frac{u_6^o(t)}{(x_6(t))^2} + C_2 \right). \quad (77)$$

4) Adjoint Boundary Conditions: The terminal and boundary conditions for the adjoint process from (16) and (17) are

$$\lambda_1(t_{s_1}) = \lambda_2(t_{s_1}+) + p_1 \quad (78)$$

$$\lambda_2(t_{s_2}) = \begin{bmatrix} r_1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_4^{(1)}(t_{s_2}+) \\ \lambda_4^{(2)}(t_{s_2}+) \end{bmatrix} + \eta_1 + 2\eta_2 x(t_{s_2}-) = r_1 \lambda_4^{(1)}(t_{s_2}+) + \eta_1 + 2\eta_2 x(t_{s_2}-) \quad (79)$$

$$\lambda_4(t_{s_3}) = \begin{bmatrix} 0 \\ r_2 \end{bmatrix} \lambda_6(t_{s_3}^+) + p_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \eta_1'' + 2\eta_2'' x(t_{s_3}^-) \end{bmatrix} = \begin{bmatrix} p_3 \\ r_2 \lambda_6(t_{s_3}+) + \eta_1'' + 2\eta_2'' x(t_{s_3}-) \end{bmatrix} \quad (80)$$

$$\lambda_6(t_f) = \alpha_1 + 2\alpha_2 x(t_f). \quad (81)$$

5) State Process: The optimal state process is obtained by the substitution of the optimal inputs (69)–(73) into the vector fields (37), (38), (40), and (42). The resulting set of differential equations, which are coupled to the adjoint dynamics (74)–(78) due to the presence of λ in the inputs (69)–(73), i.e.,

$$\dot{x}_q = \frac{\partial H_q}{\partial \lambda_q} \equiv f_q(x_q(t), u_q^o(x_q(t), \lambda_q(t))), \quad q = 1, 2, 4, 6 \quad (82)$$

and are subject to the initial and boundary conditions

$$x_1(t_0) = 0 \quad (83)$$

$$x_2(t_{s_1}) = x_1(t_{s_1}-) \quad (84)$$

$$x_4(t_{s_2}) \equiv \begin{bmatrix} x_4^{(1)}(t_{s_2}) \\ x_4^{(2)}(t_{s_2}) \end{bmatrix} = \begin{bmatrix} r_1 x_2(t_{s_2}-) \\ 0 \end{bmatrix} \quad (85)$$

$$x_6(t_{s_3}) = r_2 x_4^{(2)}(t_{s_3}-). \quad (86)$$

Moreover, the switching manifold condition must be satisfied at the autonomous switching instances t_{s_1}, t_{s_3} , i.e.,

$$x_1(t_{s_1}-) = k_1 \quad (87)$$

$$x_4^{(1)}(t_{s_3}-) = 0. \quad (88)$$

6) Hamiltonian Boundary Conditions: The Hamiltonian boundary conditions (18) at the optimal switching instances $t_{s_1}, t_{s_2}, t_{s_3}$ turn into continuity conditions

$$H_q(x_q, \lambda_q, u_q^o(x_q, \lambda_q)) = H_{q'}(x_{q'}, \lambda_{q'}, u_{q'}^o(x_{q'}, \lambda_{q'})) \\ (q, q')_{t_{s_1}} = (1, 2), \quad (q, q')_{t_{s_2}} = (2, 4), \quad (q, q')_{t_{s_3}} = (4, 6). \quad (89)$$

A detailed representation of these conditions can be found in [17].

7) Numerical Results: For the 10 (scalar) ODEs (83), (78)–(74), the three *a priori* unknown switching instances $t_{s_1}, t_{s_2}, t_{s_3}$ and the two unknown auxiliary parameters p_1, p_3 there are 15 equations provided by (84)–(87), (82)–(79), (90) in the form of initial, boundary, and terminal conditions. It is not difficult to show that for the parameter values in [15]–[17], [43], the necessary optimality conditions of the HMP in the form of the above-mentioned set of multiple-point boundary value differential equations uniquely identify optimal inputs and the corresponding optimal trajectories. The results are illustrated in

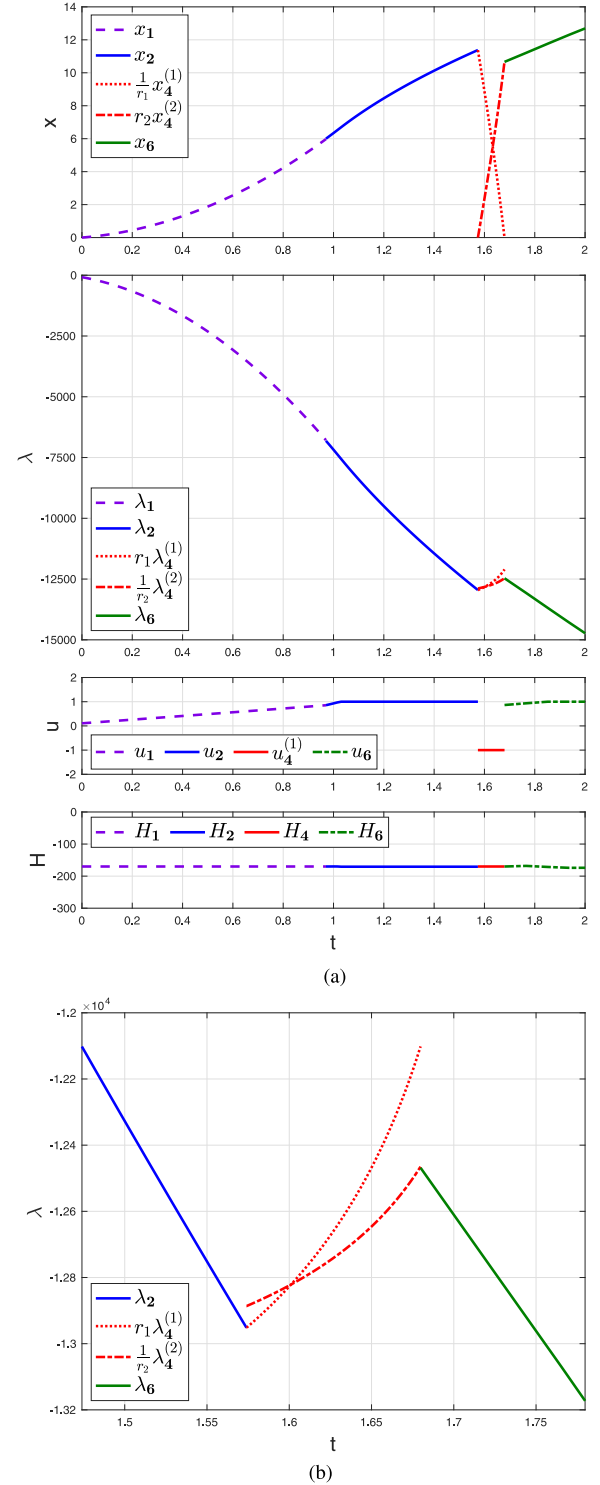


Fig. 3. HMP-based solution of the minimum energy acceleration problem for an electric vehicle with a dual planetary transmission. (a) Evolution of the (optimal) state x , adjoint process λ , optimal input u , and the Hamiltonian H over time. Due to the relatively large values of the state $x_4 = (x_4^{(1)}, x_4^{(2)})^T$, the scaled values $\frac{1}{r_1} x_4^{(1)}$ and $r_2 x_4^{(2)}$ with $r_1 = 200$ and $r_2 = 0.02$ are displayed for better illustration. (b) Satisfaction of adjoint boundary conditions (81) and (80), which are accompanied by dimension-changes. For better illustration of the equalities, state invariant switching costs $c_{2,4} = c_{4,6} = \eta_0$ are considered and instead of the adjoint components $\lambda_4 = (\lambda_4^{(1)}, \lambda_4^{(2)})^T$, the scaled values $r_1 \lambda_4^{(1)}$ and $\frac{1}{r_2} \lambda_4^{(2)}$ are displayed.

Fig. 3(a). In order to illustrate the satisfaction of the adjoint boundary conditions (81) and (80), the components $\lambda^{(1)}$ and $\lambda^{(2)}$ of the adjoint process in $t \in [t_{s2}, t_{s3}]$ are multiplied by r_1 and $\frac{1}{r_2}$, respectively, and are in-zoomed in Fig. 3(b). Interested readers are referred to [17] for more details on derivations and discussions on the results.

REFERENCES

- [1] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mishchenko, *The Mathematical Theory of Optimal Processes*, vol. 4. New York, NY, USA: Wiley, 1962.
- [2] H. Sussmann and J. Willems, "300 years of optimal control: From the brachistochrone to the maximum principle," *IEEE Control Syst.*, vol. 17, no. 3, pp. 32–44, Jun. 1997.
- [3] S. Jafarpour and A. D. Lewis, "Locally convex topologies and control theory," *Math. Control, Signals, Syst.*, vol. 28, no. 4, pp. 1–46, 2016.
- [4] F. H. Clarke and R. B. Vinter, "Optimal multiprocesses," *SIAM J. Control Optim.*, vol. 27, no. 5, pp. 1072–1091, 1989.
- [5] H. J. Sussmann, "A nonsmooth hybrid maximum principle," in *Stability and Stabilization of Nonlinear Systems*, D. Aeyels, F. Lamnabhi-Lagarigue, and A. van der Schaft, Eds., Berlin, Germany: Springer, 1999, pp. 325–354.
- [6] P. Riedinger and F. Kratz, "An optimal control approach for hybrid systems," *Eur. J. Control*, vol. 9, no. 5, pp. 449–458, 2003.
- [7] X. Xu and P. J. Antsaklis, "Optimal control of switched systems based on parameterization of the switching instants," *IEEE Trans. Autom. Control*, vol. 49, no. 1, pp. 2–16, Jan. 2004.
- [8] M. Garavello and B. Piccoli, "Hybrid necessary principle," *SIAM J. Control Optim.*, vol. 43, no. 5, pp. 1867–1887, 2005.
- [9] P. E. Caines, F. H. Clarke, X. Liu, and R. B. Vinter, "A maximum principle for hybrid optimal control problems with pathwise state constraints," in *Proc. 45th IEEE Conf. Decis. Control*, 2006, pp. 4821–4825.
- [10] M. S. Shaikh and P. E. Caines, "On the hybrid optimal control problem: Theory and algorithms," *IEEE Trans. Automat. Control*, vol. 52, no. 9, pp. 1587–1603, May 2009, corrigendum: vol. 54, no. 6, p. 1428, 2009.
- [11] V. Azhmyakov, V. Boltyanski, and A. Poznyak, "Optimal control of impulsive hybrid systems," *Nonlinear Anal., Hybrid Syst.*, vol. 2, no. 4, pp. 1089–1097, 2008.
- [12] A. V. Dmitruk and A. M. Kaganovich, "The hybrid maximum principle is a consequence of Pontryagin maximum principle," *Syst. Control Lett.*, vol. 57, no. 11, pp. 964–970, 2008.
- [13] A. V. Dmitruk and A. M. Kaganovich, "Maximum principle for optimal control problems with intermediate constraints," *Comput. Math. Model.*, vol. 22, no. 2, pp. 180–215, 2011.
- [14] F. Taringoo and P. E. Caines, "On the optimal control of impulsive hybrid systems on Riemannian manifolds," *SIAM J. Control Optim.*, vol. 51, no. 4, pp. 3127–3153, 2013.
- [15] M. S. R. Mousavi, A. Pakniyat, T. Wang, and B. Boulet, "Seamless dual brake transmission for electric vehicles: Design, control and experiment," *Mechanism Mach. Theory*, vol. 94, pp. 96–118, 2015.
- [16] B. Boulet, M. S. R. Mousavi, H. V. Alizadeh, and A. Pakniyat, "Seamless transmission systems and methods for electric vehicles," U.S. Patent 9,702,438 B2, Jul. 11, 2017.
- [17] A. Pakniyat and P. E. Caines, "Hybrid optimal control of an electric vehicle with a dual-planetary transmission," *Nonlinear Anal., Hybrid Syst.*, vol. 25, pp. 263–282, 2017.
- [18] A. Pakniyat and P. E. Caines, "Time optimal hybrid minimum principle and the gear changing problem for electric vehicles," in *Proc. 5th IFAC Conf. Anal. Des. Hybrid Syst.*, Atlanta, GA, USA, 2015, pp. 187–192.
- [19] A. Pakniyat and P. E. Caines, "On the stochastic minimum principle for hybrid systems," in *Proc. 55th IEEE Conf. Decis. Control*, Las Vegas, NV, USA, 2016, pp. 1139–1144.
- [20] D. Firoozi, A. Pakniyat, and P. E. Caines, "A mean field game - hybrid systems approach to optimal execution problems in finance with stopping times," in *Proc. 56th IEEE Conf. Decis. Control*, 2017, pp. 3144–3151.
- [21] D. Firoozi, A. Pakniyat, and P. E. Caines, "A hybrid optimal control approach to LQG mean field games with switching and stopping strategies," 2018, *arXiv:1810.02920*.
- [22] A. Pakniyat and P. E. Caines, "On the relation between the minimum principle and dynamic programming for classical and hybrid control systems," *IEEE Trans. Autom. Control*, vol. 62, no. 9, pp. 4347–4362, Sep. 2017.
- [23] A. Pakniyat and P. E. Caines, "On the relation between the minimum principle and dynamic programming for hybrid systems," in *Proc. 53rd IEEE Conf. Decis. Control*, 2014, pp. 19–24.
- [24] A. Pakniyat and P. E. Caines, "The hybrid minimum principle in the presence of switching costs," in *Proc. 52nd IEEE Conf. Decis. Control*, 2013, pp. 3831–3836.
- [25] P. E. Caines, M. Egerstedt, R. Malhamé, and A. Schöllig, "A hybrid Bellman equation for bimodal systems," in *Proc. 10th Int. Conf. Hybrid Syst., Comput. Control*, 2007, vol. 4416, pp. 656–659.
- [26] A. Schöllig, P. E. Caines, M. Egerstedt, and R. Malhamé, "A hybrid Bellman equation for systems with regional dynamics," in *Proc. 46th IEEE Conf. Decis. Control*, 2007, pp. 3393–3398.
- [27] A. Bensoussan and J. L. Menaldi, "Hybrid control and dynamic programming," *Dyn. Continuous, Discrete Impulsive Syst. Ser. B, Appl. Algorithm*, vol. 3, no. 4, pp. 395–442, 1997.
- [28] M. S. Branicky, V. S. Borkar, and S. K. Mitter, "A unified framework for hybrid control: Model and optimal control theory," *IEEE Trans. Autom. Control*, vol. 43, no. 1, pp. 31–45, Jan. 1998.
- [29] S. Dharmatti and M. Ramaswamy, "Hybrid control systems and viscosity solutions," *SIAM J. Control Optim.*, vol. 44, no. 4, pp. 1259–1288, 2005.
- [30] G. Barles, S. Dharmatti, and M. Ramaswamy, "Unbounded viscosity solutions of hybrid control systems," *ESAIM - Control, Optim. Calculus Variations*, vol. 16, no. 1, pp. 176–193, 2010.
- [31] M. Shaikh and P. Caines, "Optimality zone algorithms for hybrid systems computation and control: From exponential to linear complexity," in *Proc. 44th IEEE Conf. Decis. Control, Eur. Control Conf.*, 2005, vol. 2005, pp. 1403–1408.
- [32] F. Taringoo and P. E. Caines, "Gradient geodesic and newton geodesic HMP algorithms for the optimization of hybrid systems," *Annu. Rev. Control*, vol. 35, no. 2, pp. 187–198, 2011.
- [33] F. Taringoo and P. E. Caines, "Gradient-geodesic HMP algorithms for the optimization of hybrid systems based on the geometry of switching manifolds," in *Proc. 49th IEEE Conf. Decis. Control*, 2010, pp. 1534–1539.
- [34] H. Axelsson, Y. Wardi, M. Egerstedt, and E. Verriest, "Gradient descent approach to optimal mode scheduling in hybrid dynamical systems," *J. Optim. Theory Appl.*, vol. 136, no. 2, pp. 167–186, 2008.
- [35] M. Boccadoro, Y. Wardi, M. Egerstedt, and E. Verriest, "Optimal control of switching surfaces in hybrid dynamical systems," *Discrete Event Dyn. Syst.*, vol. 15, no. 4, pp. 433–448, 2005.
- [36] H. Gonzalez, R. Vasudevan, M. Kamgarpour, S. S. Sastry, R. Bajcsy, and C. J. Tomlin, "A descent algorithm for the optimal control of constrained nonlinear switched dynamical systems," in *Proc. 13th ACM Int. Conf. Hybrid Syst., Comput. Control*, 2010, pp. 51–60.
- [37] P. Zhao, S. Mohan, and R. Vasudevan, "Optimal control of polynomial hybrid systems via convex relaxations," *IEEE Trans. Autom. Control*, vol. 65, no. 5, pp. 2062–2077, May 2020.
- [38] B. Passenberg, M. Leibold, O. Stursberg, and M. Buss, "The minimum principle for time-varying hybrid systems with state switching and jumps," in *Proc. 50th IEEE Conf. Decis. Control, Eur. Control Conf.*, 2011, pp. 6723–6729.
- [39] R. V. Cowlagi, "Hierarchical trajectory optimization for a class of hybrid dynamical systems," *Automatica*, vol. 77, pp. 112–119, 2017.
- [40] G. Mamakoukas, M. A. MacIver, and T. D. Murphey, "Feedback synthesis for underactuated systems using sequential second-order needle variations," *Int. J. Robot. Res.*, vol. 37, no. 13/14, pp. 1826–1853, 2018.
- [41] A. Pakniyat and P. E. Caines, "On the minimum principle and dynamic programming for hybrid systems," in *Proc. 19th Int. Federation Autom. Control World Congr.*, 2014, pp. 9629–9634.
- [42] A. Pakniyat and P. E. Caines, "On the minimum principle and dynamic programming for hybrid systems with low dimensional switching manifolds," in *Proc. 54th IEEE Conf. Decis. Control*, Osaka, Japan, 2015, pp. 2567–2573.
- [43] A. Pakniyat and P. E. Caines, "On the relation between the hybrid minimum principle and hybrid dynamic programming: A linear quadratic example," in *Proc. 5th IFAC Conf. Anal. Des. Hybrid Syst.*, Atlanta, GA, USA, 2015, pp. 169–174.
- [44] P. E. Caines, "Lecture notes on nonlinear and hybrid control systems: Dynamics, stabilization and optimal control," Dept. Elect. Comput. Eng., McGill University, Montreal, QC, Canada, 2019.
- [45] A. Pakniyat, "Optimal control of deterministic and stochastic hybrid systems: Theory and applications," Ph.D. dissertation, Dept. Elect. Comput. Eng., McGill University, Montreal, QC, Canada, 2016.