On the Hybrid Minimum Principle: The Hamiltonian and Adjoint Boundary Conditions

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***Abstract*— The Hybrid Minimum Principle (HMP) is presented for the optimal control of deterministic hybrid systems with both autonomous and controlled switchings and jumps where state jumps at the switching instants are permitted to be accompanied by changes in the dimension of the state space. A feature of particular importance is the explicit presentation of the boundary conditions on the Hamiltonians and the adjoint processes before and after switchings and jumps. The numerical benefit of these expressions are demonstrated on a modified version of the Multiple Autonomous Switchings (MAS) algorithm. The results are illustrated for the hybrid model of an electric vehicle powertrain with a two-speed transmission.**

***Index Terms*— Hybrid systems, Minimum Principle, nonlinear control systems, optimal control, Pontryagin Maximum Principle**

I. Introduction

ONE of the principal approaches in solving optimal control problems is the Minimum Principle (MP), also called the Maximum Principle in the pioneering work of Pontryagin et al. [1], that provides a set of necessary conditions that must be satisfied by all optimal processes. This principle states that along optimal state processes there exist adjoint processes such that their joint governing dynamics possess a Hamiltonian canonical form and that the optimal input process is the pointwise minimizer (or the maximizer depending on the sign convention) of the Hamiltonian function. In other words, the significance of the MP is that it turns the "cost functional minimization" (over the infinite dimensional space of input processes) into a "Hamiltonian function minimization" (over the pointwise value of the input), based upon solutions of a set of two-point boundary value ordinary differential equations.

The Minimum Principle, as indicated by the name, is a *principle*, i.e. a not yet completely precise statement that requires technical conditions to be stated as a theorem [2]. For control systems with continuous dynamics these technical conditions are mostly on the regularity requirements (as, e.g., indicated in [3] they are joint conditions on [continuous] state and input processes). For hybrid control systems, however, further technical conditions need to be imposed on interactions of the continuous and discrete subsystems. Various versions of the MP for hybrid systems are available in the control theory literature [4]–[14]. However, they do not exhaust the full power of the principle. In particular, (i) the presence of both autonomous and controlled switchings, (ii) the possibility of jumps in the state at switching instances, (iii) the possibility of dimension changes in the state space, and (iv) the consideration of switching costs together with running and terminal costs, are characteristics of which only strict subsets have appeared in the literature.

However, there are several engineering systems that exhibit the features (i)–(iv) above in entirety, or with combinations for which a version of the MP is not immediately available. As an important example, one can refer to the control of electric vehicles equipped with a dual-stage planetary transmission studied in [15]–[18] whose associated hybrid optimal control problem is presented in this paper as well. Similar characteristics appear in the extention of this work to stochastic hybrid systems [19] and in Hybrid Mean Field Games theory and applications [20], [21] where an agent's state extended by the mean field terms (associated with active agents) undergoes dimension changes when a group of agents join or leave the population, and where the terminal cost of the leaving agents constitutes a switching cost for the population's mean field.

The primary objective of this paper is the presentation of a general version of the Hybrid Minimum Principle (HMP) for deterministic systems that captures all characteristics (i)–(iv) above. The regularity assumptions on the continuous dynamics are minimal and imposed primarily to ensure the existence and uniqueness of solutions as well as continuous dependence on initial conditions [22]–[24]. Further generalizations such as the lying of the system's vector fields in Riemannian spaces [14], nonsmooth assumptions [4], [5], state-dependence of the control value sets [8], and interactions with stochastic subsystems [19], as well as restrictions to certain subclasses, such as those with regional dynamics [25], [26], and with specified families of jumps [27]–[30], become possible through variations and extensions of the framework presented here.

The secondary objective of this work is the explicit expression of the boundary conditions on the Hamiltonians and adjoint processes in contrast to their implicit expressions in the literature in the form of the so-called transversality conditions. This provides a potential to improve the performance of numerical algorithms (e.g. [10], [31]–[40]) that satisfy the Hamiltonian continuity condition implicitly.

The tertiary objective of this note is to illustrate the theoretical results by means of a worked out example of energy minimization for an electric vehicle whose study requires the features (i), (ii) and (iii) above, due to the addition of a multi-speed transmission. More specifically, feature (i) is a necessity since the initiation of gear changing is a controlled switching while the termination of a gear changing process requires the satisfaction of full stop conditions for certain rotary elements, hence is an autonomous switching. Moreover, (ii) and (iii) are essential due to the possession of different mechanical degrees of freedom in each mode and the relationships between the generalized coordinates in each of those modes. Last but not least, the accommodation of (iv) permits the study of hybrid optimal control problems associated with the minimization of the total energy for the acceleration and deceleration of the vehicle with switching costs representing the energy consumption and losses contributed by the electronics operating the locks and brakes inside the transmission mechanism. Further analytic examples can be found in [41]–[43].

The organisation of the paper is as follows. The definition of hybrid systems and the associated class of hybrid optimal control problems are, respectively, presented in Sections II and III. The Hybrid Minimum Principle is presented in Section IV in the conventional Hamiltonian canonical form and with boundary conditions in the generalized transversality form. In Section V an explicit expression for the transversality conditions is presented based upon which a modified version of the Multiple Autonomous Switchings (MAS)

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algorithm is presented. The representative power of the theoretical framework and the implementation steps of the HMP results are illustrated in Section VI where the energy consumption minimization for an EV equipped with a particular transmission is studied.

II. Hybrid Systems

A (deterministic) *hybrid system* $\mathbb{H}$ is a septuple

$$\mathbb{H} = \{H, I, \Gamma, A, F, \Xi, \mathcal{M}\} \tag{1}$$

where the symbols in the expression and their governing assumptions are defined as below.

A0: $H := \coprod_{q\in Q} \mathbb{R}^{n_q}$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$, where $\coprod$ denotes disjoint union, i.e. $\coprod_{q\in Q} \mathbb{R}^{n_q} = \bigcup_{q\in Q}\left\{(q,x) : x\in\mathbb{R}^{n_q}\right\}$, where

$Q = \{\mathbf{1}, \mathbf{2}, ..., |\boldsymbol{Q}|\} \equiv \left\{q^{(1)}, q^{(2)}, ..., q^{(|Q|)}\right\}, with\, |Q| < \infty$, is a finite set of *discrete states (components)*, and

$\{\mathbb{R}^{n_q}\}_{q\in Q}$ is a family of finite dimensional continuous state spaces, where $n_q \leq n < \infty$ for all $q \in Q$.

$I := \Sigma \times U$ is the set of system input values, where

Σ with $|\Sigma| < \infty$ is the set of discrete state transition and continuous state jump events extended with the identity element,

$U = \{U_q\}_{q\in Q}$ is the set of *admissible (continuous) control values*, where each $U_q \subset \mathbb{R}^{m_q}$ is a compact set in $\mathbb{R}^{m_q}$.

The set of admissible (continuous) control inputs $\mathcal{U}(U) := L_\infty([t_0, T_*), U)$, is defined to be the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*)$, $T_* < \infty$, where the boundedness property necessarily holds here since admissible inputs take values in the compact set U.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete state transition map*.

$\Xi : H \times \Sigma \to H$ is a time independent (partially defined) *continuous state jump transition map*. For all $\xi \in \Xi$, the functions $\xi_\sigma \equiv \xi(\cdot,\sigma) : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$, $p \in A(q,\sigma)$ are assumed to be continuously differentiable in the continuous state $x \in \mathbb{R}^{n_q}$.

$A : Q \times \Sigma \to Q$ denotes both a deterministic finite automaton and the automaton's associated transition function on the state space Q and event set Σ, such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q-dependent subset $\{A(q,\sigma), \sigma\in\Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : Q \times \mathbb{R}^n \times \Sigma \to H|_Q$. In other words, Γ can only make a discrete state transition in a hybrid state (q,x) if the automaton A can make the corresponding transition in q.

F is an indexed collection of *vector fields* $\{f_q\}_{q\in Q}$ such that for each $q \in Q$ there exist $k_{f_q} \geq 1$ for which $f_q \in C^{k_{f_q}}(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}^{n_q})$ satisfies a joint uniform Lipschitz condition, i.e., there exists $L_f < \infty$ such that $\|f_q(x_1,u_1) - f_q(x_2,u_2)\| \leq L_f(\|x_1 - x_2\| + \|u_1 - u_2\|)$ for all $x, x_1, x_2 \in \mathbb{R}^{n_q}$, $u, u_1, u_2 \in U_q$.

$\mathcal{M} = \{m_\alpha : \alpha \in Q\times Q\}$ denotes a collection of *switching manifolds* such that, for any ordered pair $\alpha \equiv (\alpha_1,\alpha_2) = (q,r)$, m_α is a smooth, i.e. C^∞ codimension 1 sub-manifold of $\mathbb{R}^{n_q}$, described locally by $m_\alpha = \{x : m_\alpha(x) = 0\}$, and possibly with boundary ∂m_α. It is assumed that $m_\alpha \cap m_\beta = \emptyset$, whenever $\alpha_1 = \beta_1$ but $\alpha_2 \neq \beta_2$, for all $\alpha, \beta \in Q\times Q$. □

Switching manifolds will function in such a way that whenever a trajectory governed by the controlled vector field meets the switching manifold transversally there is an autonomous switching to another controlled vector field or there is a jump transition in the continuous state component, or both. A transversal arrival on a switching manifold $m_{q,r}$, at state $x_q \in m_{q,r} = \{x\in\mathbb{R}^{n_q} : m_{q,r}(x) = 0\}$ occurs whenever

$$\nabla m_{q,r}(x_q)^{\mathsf{T}} f_q(x_q, u_q) \neq 0, \tag{2}$$

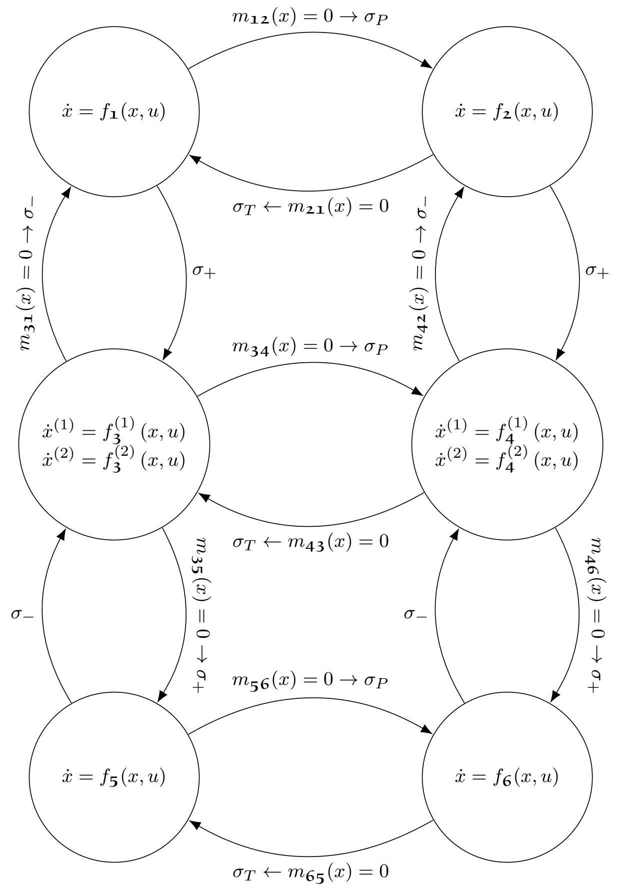


Fig. 1: The hybrid Automata diagram for the transmission-equipped electric vehicle in [15], [17] that serves as an example in Section VI.

for $u_q \in U_q$, and $q, r \in Q$. It is assumed that:

A1: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0,q_j}(x_0) \neq 0$, for all $q_j \in Q$. □

A hybrid input process defined over $[t_0, t_f), t_f < \infty$ is denoted by $I_L = (S_L, \boldsymbol{u})$, where $S_L = \left((t_0,\sigma_0), (t_1,\sigma_1), \cdots, (t_L,\sigma_L)\right)$, $L < \infty$, is a finite *hybrid sequence of switching events* with $\tau_L := \{t_0, t_1, t_2, \ldots, t_L\}$ a strictly increasing sequence of times, $\sigma_i \in \Sigma$, $i \in \{1, 2, \cdots, L\}$, with $\sigma_0 = id$, an admissible language of the automata A, and $\boldsymbol{u} \equiv \left\{\boldsymbol{u}_{q0}, \boldsymbol{u}_{q1}, \ldots, \boldsymbol{u}_{qL}\right\} \in \mathcal{U}$, with $\boldsymbol{u}_{q_i} \in L_\infty([t_i, t_{i+1}), U_{q_i})$ and admissible input. such that the associated hybrid state trajectory satisfies:

A. Continuous State Dynamics

The continuous component of the hybrid state $\boldsymbol{x} = \left\{x_{q0}(\cdot), x_{q1}(\cdot), \ldots, x_{qL}(\cdot)\right\}$ is a piecewise continuous function which is almost everywhere differentiable and on each time segment specified by τ_L satisfies the dynamical equation

$$\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \qquad a.e.\ t \in [t_i, t_{i+1}), \tag{3}$$

with the initial conditions

$$x_{q_0}(t_0) = x_0 \tag{4}$$

$$x_{q_i}(t_i) = \xi_{q_{i-1}q_i}(x_{q_{i-1}}(t_i-)) := \xi_{q_{i-1}q_i}\left(\lim_{t\uparrow t_i} x_{q_{i-1}}(t)\right) \tag{5}$$

B. Autonomous Discrete Transition Dynamics

An autonomous (uncontrolled) discrete state transition from q_{i-1} to q_i together with a continuous state jump $\xi_{q_{i-1}q_i}$ occurs at the *autonomous switching time* t_i if $x_{q_{i-1}}(t_i-) := \lim_{t \uparrow t_i} x_{q_{i-1}}(t)$ satisfies a switching manifold condition of the form

$$m_{q_{i-1}q_i}\left(x_{q_{i-1}}(t_i-), t_i\right) = 0 \tag{6}$$

for $q_i \in Q$, where $m_{q_{i-1}q_i}(x) = 0$ defines a (q_{i-1}, q_i) switching manifold and it is not the case that either (i) $x(t_i-) \in \partial m_{q_{i-1}q_i}$ or (ii) $f_{q_{i-1}}(x(t_i-), u(t_i-)) \perp \nabla m_{q_{i-1}q_i}(x(t_i-))$, i.e. t_i is not a manifold termination instant (see [44]). With the assumptions A0 and A1 in force, such a transition is well defined and labels the event $\sigma_{q_{i-1}q_i} \in \Sigma$, that corresponds to the hybrid state transition

$$\begin{aligned} h(t_i) &\equiv (q_i, x_{q_i}(t_i)) \\ &= \left(\Gamma\left(q_{i-1}, x_{q_{i-1}}(t_i-), \sigma_i\right), \xi_{q_{i-1}q_i}\left(x_{q_{i-1}}(t_i-)\right)\right) \end{aligned} \tag{7}$$

C. Controlled Discrete Transition Dynamics

A controlled discrete state transition together with a controlled continuous state jump $\xi_{q_{i-1}q_i}$ occurs at the *controlled discrete event time* t_i if t_i is not an autonomous discrete event time and if there exists a controlled discrete input event $\sigma_{q_{i-1}q_i} \in \Sigma$ for which

$$\begin{aligned} h(t_i) &\equiv (q_i, x_{q_i}(t_i)) \\ &= \left(\Gamma\left(q_{i-1}, x_{q_{i-1}}(t_i-), \sigma_i\right), \xi_{q_{i-1}q_i}\left(x_{q_{i-1}}(t_i-)\right)\right) \end{aligned} \tag{8}$$

with $\left(t_i, \sigma_{q_{i-1}q_i}\right) \in S_L$ and $q_i \in A(q_{i-1})$. □

A2: For a specified sequence of discrete states $\{q_i\}_{i=0}^{L}$, the class of admissible input-state trajectories is non-empty. □

Theorem 2.1. *[44] A hybrid system $\mathbb{H}$ with an initial hybrid state (q_0, x_0) satisfying assumptions A0, A1 and A2 possesses a unique hybrid input-state trajectory on $[t_0, T_{**})$, where T_{**} is the least of*

(i) $T_* \leq \infty$, where $[t_0, T_*)$ is the temporal domain of the definition of the hybrid system,

(ii) a manifold termination instant T_* of the trajectory $h(t) = h(t, (q_0, x_0), (S_L, u))$, $t \geq t_0$, at which either $x(T_*-) \in \partial m_{q(T_*-)q(T_*)}$ or $f_{q(T_*-)}(x(T_*-), u(T_*-)) \perp \nabla m_{q(T_*-)q(T_*)}(x(T_*-))$. □

We note that Zeno times, i.e. accumulation points of discrete transition times, are ruled out by A2.

III. Hybrid Optimal Control Problems

A3: Let $\{l_q\}_{q\in Q}$, $l_q \in C^{n_l}(\mathbb{R}^n \times U \to \mathbb{R}_+)$, $n_l \geq 1$, be a family of running cost functions; $\{c_\sigma\}_{\sigma\in\Sigma}$, $c_\sigma \in C^{n_c}(\mathbb{R}^n \times \Sigma \to \mathbb{R}_+)$, $n_c \geq 1$, be a family of switching cost functions; and $\{g_q\}_{q\in Q}$, $g_q \in C^{n_g}(\mathbb{R}^n \to \mathbb{R}_+)$, $n_g \geq 1$, be a family of terminal cost functions satisfying the following assumptions:

(i) There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $|l_q(x,u)| \leq K_l\left(1 + \|x\|^{\gamma_l}\right)$ and $|l_q(x_1,u_1) - l_q(x_2,u_2)| \leq K_l(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $q \in Q$, $x \in \mathbb{R}^{n_q}$, $u \in U_q$.

(ii) There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c_\sigma(x)| \leq K_c\left(1 + \|x\|^{\gamma_c}\right)$, $\sigma \in \Sigma$, $\sigma_1 = q$, $x \in \mathbb{R}^{n_q}$.

(iii) There exists $K_g < \infty$ and $1 \leq \gamma_g < \infty$ such that $|g_q(x)| \leq K_g\left(1 + \|x\|^{\gamma_g}\right)$, $q \in Q$, $x \in \mathbb{R}^{n_q}$. □

Consider the initial time t_0, final time $t_f < \infty$, and initial hybrid state $h_0 = (q_0, x_0)$. With the number of switchings L held fixed, the set of all hybrid input trajectories with exactly L switchings is denoted by $\boldsymbol{I_L}$. Let $I_L \in \boldsymbol{I_L}$ be a hybrid input trajectory that by Theorem 2.1 results in a unique hybrid state process. Then hybrid performance functions for the corresponding hybrid input-state trajectory are defined as

$$\begin{aligned} J(t_0, t_f, h_0, L; I_L) &:= \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s), s)\, ds \\ &+ \sum_{j=1}^{L} c_{q_{j-1}q_j}\left(t_j, x_{q_{j-1}}(t_j-)\right) + g\left(x_{q_L}(t_f)\right) \end{aligned} \tag{9}$$

IV. The Hybrid Minimum Principle (HMP)

Theorem 4.1 ([45])**.** *Consider the hybrid system $\mathbb{H}$ subject to assumptions A0-A3, and the HOCP with the hybrid performance function* (9)*. Define the family of system Hamiltonians by*

$$H_q(x_q, \lambda_q, u_q, t) = l_q(x_q, u_q, t) + \lambda_q^{\mathsf{T}} f_q(x_q, u_q, t), \tag{10}$$

$x_q, \lambda_q \in \mathbb{R}^{n_q}$, $u_q \in U_q$, $q \in Q$, and let $\{q_i\}_{i=0}^{L}$ be a specified sequence of discrete states with its associated set of switchings. Then for an optimal input u^o and along the corresponding optimal trajectory x^o, there exists an adjoint process λ^o such that

$$H_q\left(x_q^o, \lambda_q^o, u_q^o, t\right) \leq H_q\left(x_q^o, \lambda_q^o, v, t\right), \tag{11}$$

for all $v \in U_q$, where (x^o, λ^o) satisfy

$$\dot{x}_q^o = \frac{\partial H_q}{\partial \lambda_q}(x_q^o, \lambda_q^o, u_q^o, t) \equiv f_q(x_q^o, u_q^o, t), \tag{12}$$

$$\begin{aligned} \dot{\lambda}_q^o &= -\frac{\partial H_q}{\partial x_q}(x_q^o, \lambda_q^o, u_q^o, t) \\ &\equiv -\frac{\partial l_q}{\partial x}(x_q^o, u_q^o, t) - \frac{\partial f_q}{\partial x}(x_q^o, u_q^o, t)^{\mathsf{T}} \lambda_q^o, \end{aligned} \tag{13}$$

almost everywhere $t \in [t_0, t_f]$, subject to

$$x_{q_0}^o(t_0) = x_0, \tag{14}$$

$$x_{q_j}^o(t_j) = \xi_{q_{j-1}q_j}\left(x_{q_{j-1}}^o(t_j-)\right), \tag{15}$$

$$\lambda_{q_L}^o(t_f) = \nabla g\left(x_{q_L}^o(t_f)\right), \tag{16}$$

$$\begin{aligned} \lambda_{q_{j-1}}^o(t_j-) \equiv \lambda_{q_{j-1}}^o(t_j) &= \nabla \xi_{q_{j-1}q_j}^{\mathsf{T}} \lambda_{q_j}^o(t_j+) \\ &+ \nabla c_{q_{j-1}q_j} + p_j \nabla m_{q_{j-1}q_j}, \end{aligned} \tag{17}$$

where $p_j \in \mathbb{R}$ when t_j indicates the time of an autonomous switching, subject to the switching manifold condition $m_{q_{j-1}q_j}\left(x_{q_{j-1}}^o(t_j-)\right) = 0$, and $p_j = 0$ when t_j indicates the time of a controlled switching. Moreover, the Hamiltonian satisfies

$$\begin{aligned} H_{q_{j-1}}(t_j-) &\equiv H_{q_{j-1}}\left(x_{q_{j-1}}^o, \lambda_{q_{j-1}}^o, u_{q_{j-1}}^o, t_j^o-\right) \\ &= H_{q_j}\left(x_{q_j}^o, \lambda_{q_j}^o, u_{q_j}^o, t_j^o+\right) - \frac{\partial c_{q_{j-1}q_j}}{\partial t} - p_j \frac{\partial m_{q_{j-1}q_j}}{\partial t} \\ &\equiv H_{q_j}(t_j+) - \frac{\partial c_{q_{j-1}q_j}}{\partial t} - p_j \frac{\partial m_{q_{j-1}q_j}}{\partial t}. \end{aligned} \tag{18}$$

at both autonomous and controlled switching instants t_j □

V. Explicit Expressions and the Associated HMP–MAS Algorithm

Let $N := \sum_{i=0}^{L} n_{q_i}$. We remark that in the forward-backward ODEs (ordinary differential equations) (12) and (13), there is a total of $2N$ state-type components for which N initial conditions are provided by (14) and (15) and N terminal conditions are provided by (16) and (17). There are also L unknown (not apriori fixed) values for t_j for which there are L algebraic conditions provided by (18). Moreover, for every case of an autonomous switching, there is an unknown scalar p_j whose values can be uniquely determined from the simultaneous solution of the set of equations together with the set of constraints imposed by (6) whose count is the same as the number of the unknown values of p_j's. Hence, the set of necessary conditions in the HMP is complete in the sense that the number of unknowns match the number of conditions they must satisfy. However, the implicit determination of p_j's from the holistic set of differential-algebraic equations might not translate well in some numerical solution methodologies.

A. Explicit Expressions for the Hamiltonians and the Adjoints Boundary Conditions

As obtained in the authors' proof of the HMP (see e.g. [45]), the scalar p_j in the adjoint boundary conditions (17) and the Hamiltonian boundary conditions can be alternatively determined from

$$p_j := \frac{l^{q_{j-1}}_{q_j,\xi} + \lambda^{\mathsf{T}}_{q_j} f^{\xi,q_{j-1}}_{q_j,\xi} - \partial c^{1,f_{q_{j-1}}}_{t,x}}{\frac{\partial m_{q_{j-1}q_j}}{\partial t} + \left[\frac{\partial m_{q_{j-1}q_j}}{\partial x_{q_{j-1}}}\right]^T f_{q_{j-1}}} \tag{19}$$

for every autonomous switching instant t_j, where

$$l^{q_{j-1}}_{q_j,\xi} := l_{q_j}\Big(\xi(x), u\big(\xi(x), \lambda_{q_j}\big)\Big) - l_{q_{j-1}}\Big(x, u\big(x, \lambda_{q_{j-1}}\big)\Big) \tag{20}$$

$$f^{\xi,q_{j-1}}_{q_j,\xi} := f_{q_j}\Big(\xi(x), u\big(\xi(x), \lambda_{q_j}\big)\Big) - \nabla\xi f_{q_{j-1}}\Big(x, u\big(x, \lambda_{q_{j-1}}\big)\Big) \tag{21}$$

$$\partial c^{1,f_{q_{j-1}}}_{t,x} := \frac{\partial}{\partial t} c_{q_{j-1}q_j}(x) - \nabla c^{\mathsf{T}}_{q_{j-1}q_j} f_{q_{j-1}}\Big(x, u\big(x, \lambda_{q_{j-1}}\big)\Big) \tag{22}$$

and where

$$u(x,\lambda) := \operatorname*{argmin}_{u\in U_{q(t)}} H_{q(t)}\left(x, \lambda, u, t\right) \tag{23}$$

is used in equations (20), (21) and (22) for the ease of notation.

B. The HMP–MAS Algorithm

The HMP-based Multiple Autonomous Switchings (MAS) Algorithm has been proposed in [10]. This algorithm has been originally developed for a class of hybrid systems with the feature (i) mentioned in Section I, but not covering the features (ii), (iii) and (iv). In this section, we present a generalization of the HMP-MAS algorithm that covers a general class of hybrid optimal control problems with the features (i) to (iv), while also making a modification based upon the explicit expression (19) presented in the previous section.

Let $\{q_i\}_{i=0}^{L}$ be the given discrete state sequence and let $\{(\mathring{t}_i, \mathring{x}_{q_{i-1}}(\mathring{t}_i-))\}_{i=1}^{L}$ be a nominal set of feasible (but not necessarily optimal) switching times and states. By feasibility we mean that (a) for every autonomous switching pair, $(\mathring{t}_i, \mathring{x}_{q_{i-1}}(\mathring{t}_i-))$ the corresponding switching manifold condition $m_{q_{i-1}q_i}((\mathring{t}_i, \mathring{x}_{q_{i-1}}(\mathring{t}_i-))) = 0$ is satisfied, and (b) for every switching time $\mathring{t}_{i+1}$ the associated pre-switching state $\mathring{x}_{q_i}(\mathring{t}_{i+1}-))$ is reachable from the previous point, i.e. there exist some nominal $\mathring{u}_s$, $s \in [\mathring{t}_i, \mathring{t}_{i+1})$ such that

$$\mathring{x}_{q_i}\left(\mathring{t}_{i+1}-\right) = \xi_{q_{i-1}q_i}(\mathring{x}_{q_{i-1}}\left(\mathring{t}_i-\right) + \int_{\mathring{t}_i}^{\mathring{t}_{i+1}} f_{q_i}\left(\mathring{x}_s, \mathring{u}_s\right) ds \tag{24}$$

It is, however, not essential in the initiation step to generate such $\mathring{u}_s$, $s \in [\mathring{t}_i, \mathring{t}_{i+1})$ and only the reachability of $\mathring{x}_{q_i}\left(\mathring{t}_{i+1}-\right)$ from $\xi_{q_{i-1}q_i}(\mathring{x}_{q_{i-1}}\left(\mathring{t}_i-\right)$ is a sufficient information.

0) Algorithm Initiation: Fix the termination tolerance $\epsilon_f > 0$ sufficiently small, a monotonically non-decreasing sequence of step sizes $\{r_k\}$ with $r_k < 1$, and set the iteration counter $k = 0$. Set $t^k_i = \mathring{t}_i$ and $y^k_i = \mathring{x}_{q_{i-1}}(\mathring{t}_i-))$. We also use the notations $t^k_0 = t_0$, $\xi_{q_{-1}q_0}(y_0) = x_0$ and $t^k_{L+1} = t_f$.

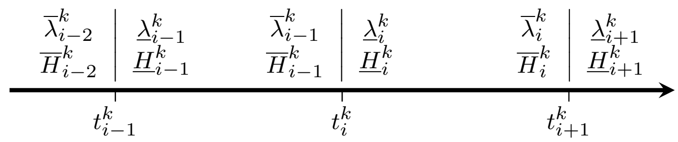


Fig. 2: Illustration of the notations used for the adjoints and the Hamiltonians at iteration k within the intervals $[t^k_i, t^k_{i+1})$.

1) Multiple Two-Point Boundary Value Problems (TPBVP): Solve the set of two-point boundary value problems (TPBVP) associated with each q_{i-1} over $[t^k_{i-1}, t^k_i)$, $i \in \{1, 2, \cdots, L\}$, with fixed initial and terminal states $\xi_{q_{i-2}q_{i-1}}(y^k_{i-1})$ and y^k_i, where these TPBVPs are decoupled in the sense that their adjoint processes and Hamiltonians are not related to each other. Obtain the initial $\lambda^k_{q_{i-1}}(t^k_{i-1})$, $H^k_{q_{i-1}}(t^k_{i-1})$ and terminal values $\lambda^k_{q_{i-1}}(t^k_i)$, $H^k_{q_{i-1}}(t^k_i)$ for the adjoints and the Hamiltonians of each of the decoupled TPBVPs that, for convenience of notation, are denoted by (see also Figure 2)

$$\underline{\lambda}^k_{i-1} := \lambda^k_{q_{i-1}}(t^k_{i-1}) \tag{25}$$

$$\overline{\lambda}^k_{i-1} := \lambda^k_{q_{i-1}}(t^k_i) \tag{26}$$

$$\underline{H}^k_{i-1} := H^k_{q_{i-1}}(t^k_{i-1}, \xi(y^k_{i-1}), u(\xi(y^k_{i-1}), \lambda^k_{q_{i-1}}(t^k_{i-1}))) \tag{27}$$

$$\overline{H}^k_{i-1} := H^k_{q_{i-1}}(t^k_i, y^k_i, u(y^k_i, \lambda^k_{q_{i-1}}(t^k_i))) \tag{28}$$

This step requires access to a classical (non-hybrid) but adjoint-based optimal control solver such as shooting-based methods.

2) Updating Procedure: Obtain new switching pairs from

$$t^{k+1}_i = t^k_i - r_k\Big(\overline{H}^k_{i-1} + p^k_i \frac{\partial m^k_i}{\partial t} - \underline{H}^k_i\Big) - r_k m^k_i \frac{\partial m^k_i}{\partial t} \tag{29}$$

$$y^{k+1}_i = y^k_i - r_k\Big(\nabla\xi^{\mathsf{T}}_{i,k}\underline{\lambda}^k_i + \nabla c^k_i + p^k_i \nabla m^k_i - \overline{\lambda}^k_{i-1}\Big) - r_k m^k_i \nabla m^k_i \tag{30}$$

where

$$p^k_i = \frac{\underline{H}^k_i - \overline{H}^k_{i-1} + f^{\mathsf{T}}_{i-1,k}\left(\overline{\lambda}^k_{i-1} - \nabla\xi^{\mathsf{T}}_{i,k}\underline{\lambda}^k_i - \nabla c^k_i\right) - \partial c_i/\partial t}{\partial m^k_i/\partial t + f^{\mathsf{T}}_{i-1,k}\nabla m^k_i} \tag{31}$$

$$f_{i-1,k} := f_{q_{i-1}}(t^k_i, y^k_i, u(y^k_i, \overline{\lambda}^k_{i-1})) \tag{32}$$

$$\xi_{i,k} := \xi_{q_{i-1}q_i}(y^k_i) \tag{33}$$

$$c^k_i := c_{q_{i-1}q_i}(y^k_i) \tag{34}$$

$$m^k_i := m_{q_{i-1}q_i}(y^k_i) \tag{35}$$

The updates (29) and (30) are descent directions for the cost

$$\mu_k = \sum_{i=1}^{L}\left\|\begin{bmatrix} \overline{H}^k_{i-1} + p^k_i\frac{\partial m^k_i}{\partial t} - \underline{H}^k_i \\ \nabla\xi^{\mathsf{T}}_{i,k}\underline{\lambda}^k_i + \nabla c^k_i + p^k_i\nabla m^k_i - \overline{\lambda}^k_{i-1}\end{bmatrix}\right\|^2 + \left|m^k_i\right|^2 \tag{36}$$

whose minimizer is a set of switching times and pre-switching states (t^*_i, y^*_i) that satisfy the HMP conditions, but at intermediate steps the switching manifold conditions might be violated. See [32] for a geodesic gradient flow algorithm that overcomes this disadvantage.

VI. Electric Vehicle with Transmission

For the illustration of the representative power of the presented HMP framework, we study an electric vehicle equipped with a dual planetary transmission presented in [15] whose hybrid systems formulation is developed in [17]. While a detailed derivation is presented in [17], in this paper the emphasise is on the control theoretic aspects of the associated hybrid optimal control problem.

A. Hybrid Systems Presentation of the Powertrain

The Automata diagram of powertrain of an EV equipped with a dual planetary transmission is illustrated in Figure 1. For this system, the discrete state set $Q = \{\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4},\mathtt{5},\mathtt{6}\}$ where the name of modes are denoted by $\mathtt{i}$ instead of q_i to avoid ambiguity when referring to q_i as the value of the discrete mode in the time interval $[t_i, t_{i+1})$. The discrete states $\mathtt{1},\mathtt{2},\mathtt{5},\mathtt{6}$ correspond to fixed gear ratios while $\mathtt{3},\mathtt{4}$ represent the system dynamics in transition between gears. The modes $\mathtt{1},\mathtt{3},\mathtt{5}$ correspond to the operation of the electric motor in low speeds where the motor torque is limited by a maximum torque constraint and the modes $\mathtt{2},\mathtt{4},\mathtt{6}$ correspond to the operation of the electric motor in high speeds where the motor torque is limited by a maximum power constraint.

During the transition between the gears, the powertrain possesses one more degree of freedom than fixed gear ratio modes, and hence the simplest hybrid state space (with the car velocity as its state in fixed gear modes and with two independent angular velocities as the state in the transition modes) gives the hybrid state space $H = \big\{(\mathtt{1},\mathbb{R}),(\mathtt{2},\mathbb{R}),(\mathtt{3},\mathbb{R}^2),(\mathtt{4},\mathbb{R}^2),(\mathtt{5},\mathbb{R}),(\mathtt{6},\mathbb{R})\big\}$.

The discrete input set $\Sigma = \{\sigma_+,\sigma_-,\sigma_P,\sigma_T\}$ where σ_+ leads the system towards a higher speed gear, σ_- directs it towars a lower speed gear, σ_P enforces the maximum power constrained operation and σ_T enforces the maximum torque constrained operation.

In the discrete states corresponding to fixed gear ratios, the normalized motor torque is the only input and hence, $U_\mathtt{1} = U_\mathtt{2} = U_\mathtt{5} = U_\mathtt{6} = [-1,1] \subset \mathbb{R}$, where negative values correspond to the regeneration mode of the electric motor. In the discrete states corresponding to transition phases, the normalized forces of the two brakes operating the transmission are inputs, in addition to the motor torque and hence, $U_\mathtt{3} = U_\mathtt{4} = [-1,1]\times[0,1]\times[0,1] \subset \mathbb{R}^3$ where, obviously, the last two components of the input corresponding to the brake forces do not change signs.

The discrete state transition map Γ and the finite automaton A are illustrated in Figure 1. As can be observed in this figure, Γ can only make a discrete state transition in a hybrid state (q,x) if the automaton A can make the corresponding transition in q.

The elements in the set of vector fields F are given as

$$f_\mathtt{1}(x,u) = -A_1 x^2 + B_1 u - C_1 x - D_1\,, \tag{37}$$

$$f_\mathtt{2}(x,u) = -A_1 x^2 + B_1 \frac{u}{x} - C_1 x - D_1\,, \tag{38}$$

$$\begin{aligned} f_\mathtt{3}^{(1)}(x,u) &= -A_{ss}x^{(1)} + A_{sr}x^{(2)} - A_{sa}\left(x^{(1)} + R_2 x^{(2)}\right)^2 \\ &\quad + B_{smt}u^{(1)} + B_{sst}u^{(2)} - B_{srt}u^{(3)} - D_{sl}\,, \\ f_\mathtt{3}^{(2)}(x,u) &= A_{rs}x^{(1)} - A_{rr}x^{(2)} - A_{ra}\left(x^{(1)} + R_2 x^{(2)}\right)^2 \\ &\quad + B_{rmt}u^{(1)} - B_{rst}u^{(2)} - B_{rrt}u^{(3)} - D_{rl}\,, \end{aligned} \tag{39}$$

$$\begin{aligned} f_\mathtt{4}^{(1)}(x,u) &= -A_{ss}x^{(1)} + A_{sr}x^{(2)} - A_{sa}\left(x^{(1)} + R_2 x^{(2)}\right)^2 \\ &\quad + B_{sm}\frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} + B_{ss}u^{(2)} - B_{sr}u^{(3)} - D_{sl}\,, \\ f_\mathtt{4}^{(2)}(x,u) &= A_{rs}x^{(1)} - A_{rr}x^{(2)} - A_{ra}\left(x^{(1)} + R_2 x^{(2)}\right)^2 \\ &\quad + B_{rm}\frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} - B_{rs}u^{(2)} + B_{rr}u^{(3)} - D_{rl}\,, \end{aligned} \tag{40}$$

$$f_\mathtt{5}(x,u) = -A_2 x^2 + B_2 u - C_2 x - D_2\,, \tag{41}$$

$$f_\mathtt{6}(x,u) = -A_2 x^2 + B_2 \frac{u}{x} - C_2 x - D_2\,, \tag{42}$$

where the coefficients $A_\bullet$, $B_\bullet$, $C_\bullet$, $D_\bullet$ and $R_\bullet$ in the above equations and $r_\bullet$ and $k_\bullet$ in the following equations are model parameters whose numerical values can be found in [17].

The set of jump transition maps Ξ is identified by

$$\xi_{\mathtt{12}} = \xi_{\mathtt{21}} = \mathrm{id}_\mathbb{R} :\ x \to x\,, \tag{43}$$

$$\xi_{\mathtt{13}} = \xi_{\mathtt{24}} :\ x \to \begin{bmatrix} r_1 x \\ 0 \end{bmatrix}, \tag{44}$$

$$\xi_{\mathtt{31}} = \xi_{\mathtt{42}} :\ \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \to \frac{x^{(1)}}{r_1}\,, \tag{45}$$

$$\xi_{\mathtt{34}} = \xi_{\mathtt{43}} = \mathrm{id}_{\mathbb{R}^2} :\ \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \to \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}, \tag{46}$$

$$\xi_{\mathtt{35}} = \xi_{\mathtt{46}} :\ \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \to r_2 x^{(2)}\,, \tag{47}$$

$$\xi_{\mathtt{53}} = \xi_{\mathtt{64}} :\ x \to \begin{bmatrix} 0 \\ \frac{x}{r_2} \end{bmatrix}, \tag{48}$$

$$\xi_{\mathtt{56}} = \xi_{\mathtt{65}} = \mathrm{id}_\mathbb{R} :\ x \to x\,, \tag{49}$$

While initiations of gear changing can be made freely (and therefore switchings to $\mathtt{3},\mathtt{4}$ are controlled), the transitions back to a fixed gear mode require the full stop for one of the degrees of freedom. Moreover, switchings between torque-constrained and power-constrained modes occur whenever the motor speed reaches a certain value. The set of switching manifolds $\mathcal{M}$ for the autonomous switchings are given by

$$m_{\mathtt{12}} = m_{\mathtt{21}} \equiv \Big\{x \in \mathbb{R} : x - k_1 = 0\Big\} \cup \Big\{x \in \mathbb{R} : x + k_1 = 0\Big\}\,, \tag{50}$$

$$m_{\mathtt{31}} = m_{\mathtt{42}} \equiv \Big\{x \in \mathbb{R}^2 : \begin{bmatrix} 0 & 1 \end{bmatrix} x = 0\Big\}\,, \tag{51}$$

$$\begin{aligned} m_{\mathtt{34}} = m_{\mathtt{43}} &\equiv \Big\{x \in \mathbb{R}^2 : \begin{bmatrix} 1 & R_1 \end{bmatrix} x - k_2 = 0\Big\} \\ &\quad \cup \Big\{x \in \mathbb{R}^2 : \begin{bmatrix} 1 & R_1 \end{bmatrix} x + k_2 = 0\Big\}\,, \end{aligned} \tag{52}$$

$$m_{\mathtt{35}} = m_{\mathtt{46}} \equiv \Big\{x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \end{bmatrix} x = 0\Big\}\,, \tag{53}$$

$$m_{\mathtt{56}} = m_{\mathtt{65}} \equiv \Big\{x \in \mathbb{R} : x - k_3 = 0\Big\} \cup \Big\{x \in \mathbb{R} : x + k_3 = 0\Big\}\,, \tag{54}$$

B. Association of Costs

Depending on the goal, one can associate numerous optimal control problems for the powertrain. For time optimal tasks, the running costs shall be taken to be $l_q(x,u) = 1$ for all $q \in Q \equiv \{\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4},\mathtt{5},\mathtt{6}\}$, so that once integrated, their sum gives the total spent time (see [17], [18] for examples of this class). For the minimization of energy consumption, the running costs shall be taken to be the power consumption rates that are determined from the motor efficiency map (see [17] for the derivation and more discussion). The resulting expressions for l_q's are presented below.

$$l_\mathtt{1}(x,u) = a_1 u^2 + b_1 x u + c_1 u + d_1 x\,, \tag{55}$$

$$l_\mathtt{2}(x,u) = a_1 \frac{u^2}{x^2} + b_1 u + c_1 \frac{u}{x} + d_1 x\,, \tag{56}$$

$$\begin{aligned} l_\mathtt{3}(x,u) &= a_{tr}\big(u^{(1)}\big)^2 + b_{tr}u^{(1)}\big(x^{(1)} + R_1 x^{(2)}\big) \\ &\quad + c_{tr}u^{(1)} + d_{tr}\big(x^{(1)} + R_1 x^{(2)}\big)\,, \end{aligned} \tag{57}$$

$$\begin{aligned} l_\mathtt{4}(x,u) &= a_{tr}\frac{\big(u^{(1)}\big)^2}{\big(x^{(1)} + R_1 x^{(2)}\big)^2} + b_{tr}u^{(1)} \\ &\quad + c_{tr}\frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} + d_{tr}\big(x^{(1)} + R_1 x^{(2)}\big)\,, \end{aligned} \tag{58}$$

$$l_\mathtt{5}(x,u) = a_2 u^2 + b_2 u + c_2 x + d_2 x\,, \tag{59}$$

$$l_\mathtt{6}(x,u) = a_2 \frac{u^2}{x^2} + b_2 u + c_2 \frac{u}{x} + d_2 x\,. \tag{60}$$

For time optimal goals, one can associate (potentially state-dependent) switching costs by considering unaccounted delays in switching. For energy optimal goals, switching costs represent the energy consumption of the mechanism performing the engagement and release of the lock holding the stationary parts fixed. While the HMP framework permits a wide range of nonlinear costs that depend on the rotational speed of transmission elements (gears), we consider a quadratic model fit, i.e.

$$c_{\mathbf{13}}(x) = c_{\mathbf{24}}(x) = c_{\mathbf{53}}(x) = c_{\mathbf{64}}(x) = \eta_0 + \eta_1 x + \eta_2 x^2, \quad (61)$$

$$c_{\mathbf{31}}(x) = c_{\mathbf{42}}(x) = \eta_0' + \eta_1' x^{(1)} + \eta_2' (x^{(1)})^2, \quad (62)$$

$$c_{\mathbf{35}}(x) = c_{\mathbf{46}}(x) = \eta_0'' + \eta_1'' x^{(2)} + \eta_2'' (x^{(2)})^2, \quad (63)$$

Notice that the switching cost in (62) contains only the first component of the state $x^{(1)}$ because the second component $x^{(2)}$ corresponds to the speed of the common ring gear in the transmission vanishes as it needs to come to a full stop for the switching to occur. Similarly, (63) contains only the second component of the state $x^{(2)}$ because the common sun gear needs to come to a full stop at these switching instances. For the numerical simulations in this section, we assume a quadratic representation for the terminal cost

$$g\left(x\left(t_f\right)\right) = \alpha_0 + \alpha_1 x\left(t_f\right) + \alpha_2 x\left(t_f\right)^2, \quad (64)$$

while any nonlinear representation satisfying A3 is also permitted.

C. The HMP Formulation

We consider the energy consumption minimization for acceleration in the first gear from the stationary state, i.e. $h_0 \equiv (q, x)(t_0) = (\mathbf{1}, 0)$ for a period of 2 seconds, i.e. $[t_0, t_f] = [0, 2]$. We would like. to perform a gear change to the second gear ratio, that corresponds to $\mathbf{6}$ since we know the vehicle's speed exceeds $12\,\mathrm{m/s} = 43.2\mathrm{km/h}$ that results in the operation of the electric motor in the power constrained region. Based upon the Hybrid Automata in Figure 1, potential sequences of length $L = 4$ for the discrete state are $\mathbf{1356}$, $\mathbf{1246}$, $\mathbf{1346}$. Thus, the HMP needs to be solved three times (one for each sequence) in order to determine the global optimal sequence together with the associated inputs. It turns out that (see [17]) that $\mathbf{1246}$ is the optimal sequence and hence, we present the analysis only for this sequence, while the results for other sequences follow the same derivation.

1) Formation of the Hamiltonians: The family of system Hamiltonians are formed as

$$H_{\mathbf{1}}(x, \lambda, u) = \lambda\left(-A_1 x^2 + B_1 u - C_1 x - D_1\right) + a_1 u^2 + b_1 x u + c_1 u + d_1 x\,, \quad (65)$$

$$H_{\mathbf{2}}(x, \lambda, u) = \lambda\left(-A_1 x^2 + B_1 \frac{u}{x} - C_1 x - D_1\right) + a_1 \frac{u^2}{x^2} + b_1 u + c_1 \frac{u}{x} + d_1 x\,, \quad (66)$$

$$\begin{aligned} H_{\mathbf{4}}(x, \lambda, u) = &\Bigg(- A_{ss} x^{(1)} + A_{sr} x^{(2)} - A_{sa}\left(x^{(1)} + R_2 x^{(2)}\right)^2 \\ &+ B_{sm} \frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} + B_{ss} u^{(2)} - B_{sr} u^{(3)} - D_{sl}\Bigg)\lambda^{(1)} \\ &+ \Bigg(A_{rs} x^{(1)} - A_{rr} x^{(2)} - A_{ra}\left(x^{(1)} + R_2 x^{(2)}\right)^2 \\ &+ B_{rm} \frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} - B_{rs} u^{(2)} + B_{rr} u^{(3)} - D_{rl}\Bigg)\lambda^{(2)} \\ &+ a_{tr} \frac{\left(u^{(1)}\right)^2}{\left(x^{(1)} + R_1 x^{(2)}\right)^2} + b_{tr} u^{(1)} + c_{tr} \frac{u^{(1)}}{x^{(1)} + R_1 x^{(2)}} \\ &+ d_{tr}\left(x^{(1)} + R_1 x^{(2)}\right)\,, \end{aligned} \quad (67)$$

$$H_{\mathbf{6}}(x, \lambda, u) = \lambda\left(-A_2 x^2 + B_2 \frac{u}{x} - C_2 x - D_2\right) + a_2 \frac{u^2}{x^2} + b_2 u + c_2 \frac{u}{x} + d_2 x\,. \quad (68)$$

2) Hamiltonian Minimization:: The Hamiltonian minimization condition (11) for the Hamiltonians (65)–(68) determines the optimal inputs as

$$u_{\mathbf{1}}(t) = \operatorname*{sat}_{[-1,1]}\left(\frac{-\left(b_1 x(t) + c_1 + B_1 \lambda(t)\right)}{2a_1}\right), \quad (69)$$

$$u_{\mathbf{2}}(t) = \operatorname*{sat}_{[-1,1]}\left(\frac{-x(t)\left(b_1 x(t) + c_1 + B_1 \lambda(t)\right)}{2a_1}\right), \quad (70)$$

$$\begin{aligned} u_{\mathbf{4}}^{(1)}(t) &= \operatorname*{sat}_{[-1,1]}\left(\frac{-X_M(t)\left[b_{tr} X_M(t) + c_{tr} + B_{sm}\lambda^{(1)}_{(t)} + B_{rm}\lambda^{(2)}_{(t)}\right]}{2a_{tr}}\right), \\ u_{\mathbf{4}}^{(2)}(t) &= \begin{cases} -1 & \text{if} \quad B_{ss}\lambda^{(1)}(t) - B_{rs}\lambda^{(2)}(t) \geq 0 \\ 0 & \text{if} \quad B_{ss}\lambda^{(1)}(t) - B_{rs}\lambda^{(2)}(t) < 0 \end{cases}, \\ u_{\mathbf{4}}^{(3)}(t) &= \begin{cases} -1 & \text{if} \quad B_{rr}\lambda^{(2)}(t) - B_{sr}\lambda^{(1)}(t) \geq 0 \\ 0 & \text{if} \quad B_{rr}\lambda^{(2)}(t) - B_{sr}\lambda^{(1)}(t) < 0 \end{cases}, \end{aligned} \quad (71)$$

$$u_{\mathbf{6}}(t) = \operatorname*{sat}_{[-1,1]}\left(\frac{-x(t)\left(b_2 x(t) + c_2 + B_2 \lambda(t)\right)}{2a_2}\right), \quad (72)$$

where $x(t) \equiv x^o_{q_i}(t)$ and $X_M(t) := \left(x^{(1)}_{(t)} + R_1 x^{(2)}_{(t)}\right)$ are employed to shorten the notation.

3) Adjoint Dynamics: The dynamics of the (backward) adjoint processes are derived from (13) as

$$\dot{\lambda}_{\mathbf{1}} = -b_1 u^o_{\mathbf{1}}(t) - d_1 + \lambda_{\mathbf{1}}(t)\left(2A_1 x_{\mathbf{1}}(t) + C_1\right) \quad (73)$$

$$\dot{\lambda}_{\mathbf{2}} = \frac{2a_1\left(u^o_{\mathbf{2}}(t)\right)^2}{\left(x_{\mathbf{2}}(t)\right)^3} + \frac{c_1 u^o_{\mathbf{2}}(t)}{\left(x_{\mathbf{2}}(t)\right)^2} - d_1 + \lambda_{\mathbf{2}}(t)\left(2A_1 x_{\mathbf{2}}(t) + B_1 \frac{u^o_{\mathbf{2}}(t)}{\left(x_{\mathbf{2}}(t)\right)^2} + C_1\right) \quad (74)$$

$$\begin{aligned} \dot{\lambda}_{\mathbf{4}}^{(1)} = &\frac{2a_{tr}\left(u_{\mathbf{4}}^{o(1)}(t)\right)^2}{\left(x^{(1)} + R_1 x^{(2)}\right)^3} + \frac{c_{tr} u_{\mathbf{4}}^{o(1)}(t)}{\left(x^{(1)} + R_1 x^{(2)}\right)^2} - d_{tr} \\ &+ \lambda_{\mathbf{4}}^{(1)}\left(A_{ss} + 2A_{sa}\left(x^{(1)} + R_2 x^{(2)}\right) + \frac{B_{sm} u_{\mathbf{4}}^{o(1)}}{\left(x^{(1)} + R_1 x^{(2)}\right)^2}\right) \\ &+ \lambda_{\mathbf{4}}^{(2)}\left(-A_{rs} + 2A_{ra}\left(x^{(1)} + R_2 x^{(2)}\right) + \frac{B_{rm} u_{\mathbf{4}}^{o(1)}}{\left(x^{(1)} + R_1 x^{(2)}\right)^2}\right) \end{aligned} \quad (75)$$

$$\begin{aligned} \dot{\lambda}_{\mathbf{4}}^{(2)} = &\left(\frac{2R_1 a_{tr}\left(u_{\mathbf{4}}^{o(1)}(t)\right)^2}{\left(x^{(1)} + R_1 x^{(2)}\right)^3} + \frac{R_1 c_{tr} u_{\mathbf{4}}^{o(1)}(t)}{\left(x^{(1)} + R_1 x^{(2)}\right)^2} - R_1 d_{tr}\right) \\ &+ \lambda_{\mathbf{4}}^{(1)}\left(-A_{sr} + 2R_2 A_{sa}\left(x^{(1)} + R_2 x^{(2)}\right) + \frac{R_1 B_{sm} u_{\mathbf{4}}^{o(1)}}{\left(x^{(1)} + R_1 x^{(2)}\right)^2}\right) \\ &+ \lambda_{\mathbf{4}}^{(2)}\left(A_{rr} + 2R_2 A_{ra}\left(x^{(1)} + R_2 x^{(2)}\right) + \frac{R_1 B_{rm} u_{\mathbf{4}}^{o(1)}}{\left(x^{(1)} + R_1 x^{(2)}\right)^2}\right) \end{aligned} \quad (76)$$

$$\dot{\lambda}_{\mathbf{6}} = \frac{2a_2\left(u^o_{\mathbf{6}}(t)\right)^2}{\left(x_{\mathbf{6}}(t)\right)^3} + \frac{c_2 u^o_{\mathbf{6}}(t)}{\left(x_{\mathbf{6}}(t)\right)^2} - d_2 + \lambda_{\mathbf{6}}(t)\left(2A_2 x_{\mathbf{6}}(t) + B_2 \frac{u^o_{\mathbf{6}}(t)}{\left(x_{\mathbf{6}}(t)\right)^2} + C_2\right) \quad (77)$$

4) Adjoint Boundary Conditions: The terminal and boundary conditions for the adjoint process from (16) and (17) are

$$\lambda_{\mathbf{1}}\left(t_{s_1}\right)=\lambda_{\mathbf{2}}\left(t_{s_1}+\right)+p_1\,. \tag{78}$$

$$\begin{aligned}\lambda_{\mathbf{2}}\left(t_{s_2}\right)&=\left[\begin{array}{cc} r_1 & 0\end{array}\right]\left[\begin{array}{c}\lambda_{\mathbf{4}}^{(1)}\left(t_{s_2}+\right)\\ \lambda_{\mathbf{4}}^{(2)}\left(t_{s_2}+\right)\end{array}\right]+\eta_1+2\eta_2 x\left(t_{s_2}-\right)\\ &=r_1\lambda_{\mathbf{4}}^{(1)}\left(t_{s_2}+\right)+\eta_1+2\eta_2 x\left(t_{s_2}-\right)\,,\end{aligned} \tag{79}$$

$$\begin{aligned}\lambda_{\mathbf{4}}\left(t_{s_3}\right)&=\left[\begin{array}{c}0\\ r_2\end{array}\right]\lambda_{\mathbf{6}}\left(t_{s_3}^{+}\right)+p_3\left[\begin{array}{c}1\\ 0\end{array}\right]+\left[\begin{array}{c}0\\ \eta_1''+2\eta_2'' x\left(t_{s_3}^{-}\right)\end{array}\right]\\ &=\left[\begin{array}{c}p_3\\ r_2\lambda_{\mathbf{6}}\left(t_{s_3}+\right)+\eta_1''+2\eta_2'' x\left(t_{s_3}-\right)\end{array}\right],\end{aligned} \tag{80}$$

$$\lambda_{\mathbf{6}}\left(t_f\right)=\alpha_1+2\alpha_2 x\left(t_f\right)\,, \tag{81}$$

5) State Process: The optimal state process is obtained by the substitution of the optimal inputs (69) to (72) into the vector fields (37), (38), (40) and (42). The resulting set of differential equations, that are coupled to the adjoint dynamics (73) to (77) due to the presence of λ in the inputs (69) to (72), i.e.

$$\dot{x}_q=\frac{\partial H_q}{\partial \lambda_q}\equiv f_q\big(x_q(t),u_q^o(x_q(t),\lambda_q(t))\big),\qquad q=\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{6} \tag{82}$$

and are subject to the initial and boundary conditions:

$$x_{\mathbf{1}}\left(t_0\right)=0\,, \tag{83}$$

$$x_{\mathbf{2}}\left(t_{s_1}\right)=x_{\mathbf{1}}\left(t_{s_1}-\right)\,, \tag{84}$$

$$x_{\mathbf{4}}\left(t_{s_2}\right)\equiv\left[\begin{array}{c}x_{\mathbf{4}}^{(1)}\left(t_{s_2}\right)\\ x_{\mathbf{4}}^{(2)}\left(t_{s_2}\right)\end{array}\right]=\left[\begin{array}{c}r_1 x_{\mathbf{2}}\left(t_{s_2}-\right)\\ 0\end{array}\right], \tag{85}$$

$$x_{\mathbf{6}}\left(t_{s_3}\right)=r_2\, x_{\mathbf{4}}^{(2)}\left(t_{s_3}-\right)\,, \tag{86}$$

Moreover, the switching manifold condition must be satisfied at the autonomous switching instances t_{s_1}, t_{s_3}, i.e.

$$x_{\mathbf{1}}\left(t_{s_1}-\right)=k_1\,, \tag{87}$$

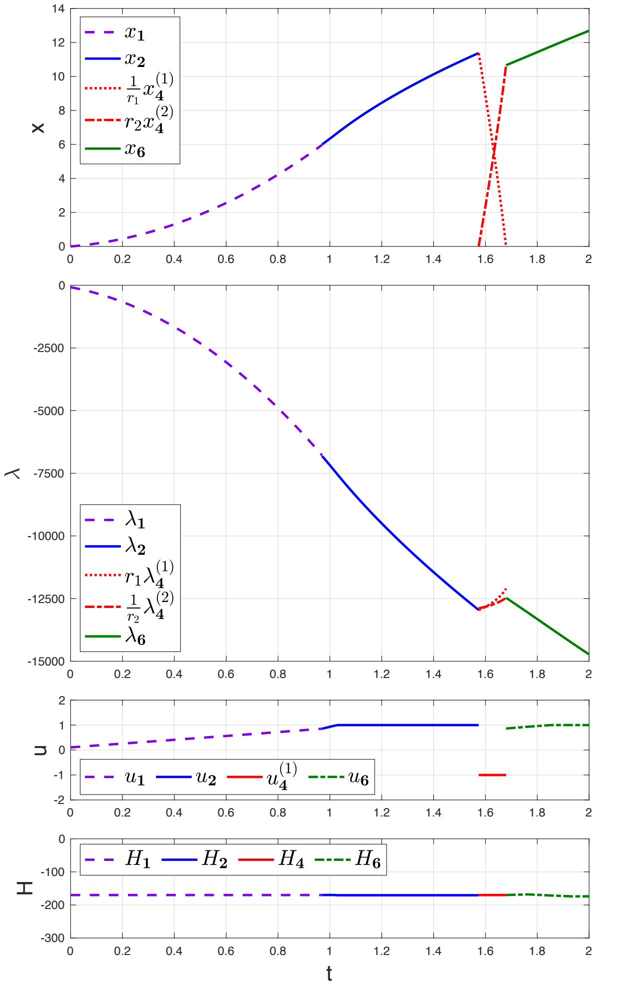
$$x_{\mathbf{4}}^{(1)}\left(t_{s_3}-\right)=0\,. \tag{88}$$

6) Hamiltonian Boundary Conditions: The Hamiltonian boundary conditions (18) at the optimal switching instances t_{s_1}, t_{s_2}, t_{s_3} turn into continuity conditions

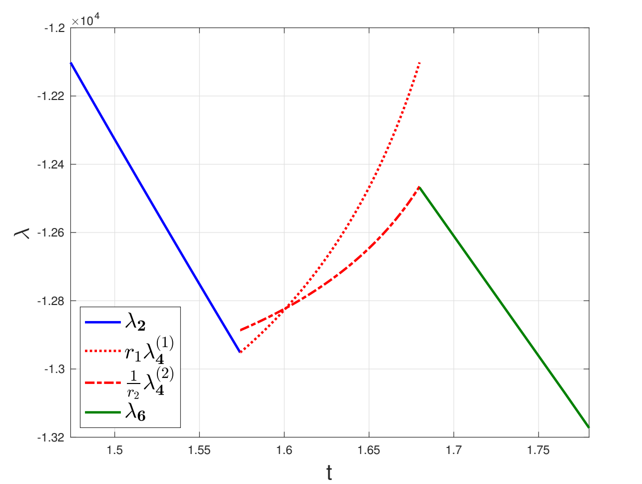
$$\begin{aligned}&H_q\left(x_q,\lambda_q,u_q^o\left(x_q,\lambda_q\right)\right)=H_{q'}\left(x_{q'},\lambda_{q'},u_{q'}^o\left(x_{q'},\lambda_{q'}\right)\right),\\ &(q,q')_{t_{s_1}}=(\mathbf{1},\mathbf{2}),\quad (q,q')_{t_{s_2}}=(\mathbf{2},\mathbf{4}),\quad (q,q')_{t_{s_3}}=(\mathbf{4},\mathbf{6}).\end{aligned} \tag{89}$$

A detailed representation of these conditions can be found in [17].

7) Numerical Results: For the 10 (scalar) ordinary differential equations (82), (77)–(73), the 3 a priori unknown switching instances t_{s_1}, t_{s_2}, t_{s_3} and the 2 unknown auxiliary parameters p_1, p_3 there are 15 equations provided by (83)–(86), (81)–(78), (89) in the form of initial, boundary and terminal conditions. It is not difficult to show that for the parameter values in [15]–[17], [43], the necessary optimality conditions of the HMP in the form of the above set of multiple-point boundary value differential equations uniquely identify optimal inputs and the corresponding optimal trajectories. The results are illustrated in Figure 3 (a). In order to illustrate the satisfaction of the adjoint boundary conditions (80) and (79), the components $\lambda^{(1)}$ and $\lambda^{(2)}$ of the adjoint process in $t\in[t_{s_2},t_{s_3}]$ are multiplied by r_1 and $\frac{1}{r_2}$ respectively, and are in-zoomed in Figure 3 (b). Interested readers are referred to [17] for more details on derivations and discussions on the results.



(a) The evolution of the (optimal) state x, adjoint process λ, optimal input u and the Hamiltonian H over time. Due to the relatively large values of the state $x_4=(x_4^{(1)},x_4^{(2)})^{\mathsf{T}}$, the scaled values $\frac{1}{r_1}x_4^{(1)}$ and $r_2x_4^{(2)}$ with $r_1=200$ and $r_2=0.02$ are displayed for better illustration.



(b) Satisfaction of adjoint boundary conditions (80) and (79) which are accompanied by dimension-changes. For better illustration of the equalities, state invariant switching costs $c_{24}=c_{46}=\eta_0$ are considered and instead of the adjoint components $\lambda_4=(\lambda_4^{(1)},\lambda_4^{(2)})^{\mathsf{T}}$, the scaled values $r_1\lambda_4^{(1)}$ and $\frac{1}{r_2}\lambda_4^{(2)}$ are displayed.

Fig. 3: HMP-based solution of the minimum energy acceleration problem for an electric vehicle with a dual planetary transmission.

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