On the Relation Between the Minimum Principle and Dynamic Programming for Classical and Hybrid Control Systems

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***Abstract*—Hybrid optimal control problems are studied for a general class of hybrid systems, where autonomous and controlled state jumps are allowed at the switching instants, and in addition to terminal and running costs, switching between discrete states incurs costs. The statements of the Hybrid Minimum Principle and Hybrid Dynamic Programming are presented in this framework, and it is shown that under certain assumptions, the adjoint process in the Hybrid Minimum Principle and the gradient of the value function in Hybrid Dynamic Programming are governed by the same set of differential equations and have the same boundary conditions and hence are almost everywhere identical to each other along optimal trajectories. Analytic examples are provided to illustrate the results.**

***Index Terms*—Dynamic programming (DP), Hamilton-Jacobi-Bellman equation, hybrid systems, nonlinear control system, optimal control, Pontryagin minimum principle (MP).**

I. Introduction

Pontryagin's minimum principle (MP) [1] and Bellman's dynamic programming (DP) [2] serve as the two key tools in optimal control theory. The MP can be considered as a generalization of Hamilton's canonical system in classical mechanics as well as an extension of Weierstrass's necessary conditions in calculus of variations [3], [4], while DP theory, including the Hamilton–Jacobi–Bellman (HJB) equation, may be considered as an extension of the Hamilton–Jacobi theory in classical mechanics and of Carathéodory's theorem in calculus of variations [5]. Both the MP and DP constitute necessary conditions for optimality, which under certain assumptions become sufficient (see, e.g., [5]–[9]). However, DP is widely used as a set of sufficient optimality conditions after the optimal control extension of Carathéodory's sufficient conditions in calculus of variations (see, e.g., [5]–[7]).

The relationship between the MP and DP, which were developed independently in 1950s, was addressed as early as the formal announcement of the Pontryagin MP [1]. In the classical optimal control framework, this relationship has been elaborated by many others since then (see, e.g., [5]–[16]). The result states that, under certain assumptions (see, e.g., [6] and [8]), the adjoint process in the MP and the gradient of the value function in DP are equal, a property which we shall sometimes refer to as the adjoint–gradient relationship. While this relationship has been proved in various forms, the majority of arguments are based on the following two key elements: (i) the assumption of the openness of the set of all points, from which an optimal transition to the reference trajectory is possible [1, p. 70]; and (ii) the inference of the extremality of the reference optimal state for the corresponding optimal control [1, p. 72]. Then, with the assumption of twice continuous differentiability of the value function, the method of characteristics (see, e.g., [6] and [8]) can be employed to obtain the aforementioned relationship, which is analogous to the derivation of the equivalence of the Hamiltonian system and the Hamilton–Jacobi equation. For certain classes of optimal control problems, the assumption of twice differentiability is intrinsically satisfied since the total cost can become arbitrarily large and negative if the second partial derivative ceases to exist (see, e.g., [16]). But, in general, even once differentiability of the value function is violated at certain points for numerous problems (see, e.g., [6], [7], and [15]–[18]). Consequently, the adjoint–gradient relationship is usually expressed within the general framework of nonsmooth analysis that declares the inclusion of the adjoint process in the set of generalized gradients of the value function [10]–[15]. However, the general expression of the adjoint–gradient relationship in the framework of nonsmooth analysis is unnecessary for optimal control problems with appropriately smooth vector fields and costs, when the optimal feedback control possesses an admissible set of discontinuities [8].

In contrast to classical optimal control theory, the relation between the MP and DP in the hybrid system framework has been the subject of limited number of studies (see, e.g., [19]–[21]). One of the main difficulties in this discussion is that the domains of definition of hybrid systems employed for the derivation of the results of hybrid optimal control theory (see, e.g., [22]–[50]) do not necessarily intersect in a general class of systems. This is especially due to the difference in the approach and the assumptions required for the derivation of necessary and sufficient optimality conditions in the two key formulations. Hence, in order to establish the relationship between the Hybrid Minimum Principle (HMP) and Hybrid Dynamic Programming (HDP), there is the preliminary task of choosing an appropriately general class of hybrid systems, within which the desired HMP

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and HDP results are valid. This is one of the tasks addressed in this paper before establishing the relation between the MP and DP for classical and hybrid control systems.

The organization of this paper is as follows. In Section II, a definition of hybrid systems is presented that covers a general class of nonlinear systems on Euclidean spaces with autonomous and controlled switchings and jumps allowed at the switching states and times. Further generalizations such as the lying of the system's vector fields in Riemannian spaces [29], [30], nonsmooth assumptions [22]–[25], [38], [39], and state dependence of the control value sets [31], as well as restrictions to certain subclasses such as those with regional dynamics [43], [44], specified families of jumps [38]–[41], and systems with locally controllable trajectories [28], which are imposed for the sufficiency of the results, are avoided so that for the selected class of hybrid systems, the associated HMP and HDP theory, together with their relationships, are derived in a unified general framework. Similarly, the selected class of hybrid optimal control problems in Section III covers a general class of hybrid optimal control problems with a large range of running, terminal, and switching costs. With the exception of the infinite horizon problems considered in [37]–[41], this framework is in accordance with the majority of the work on the HMP (see [19]–[21] and [24]–[34]) and a number of publications on HDP (see, e.g., [42]–[45]) defined on finite horizons.

The statement of the HMP is presented in Section IV, where necessary conditions are provided for the optimality of the trajectory and the controls of a hybrid system with fixed initial conditions and a sequence of autonomous or controlled switchings. These conditions are expressed in terms of the minimization of the distinct Hamiltonians indexed by the discrete-state sequence of the hybrid trajectory. A feature of special interest is the set of boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization to the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations [51].

In Section V, the principle of optimality is employed to introduce the cost-to-go and the value functions. It is proved that on a bounded set in the state space, the cost-to-go functions are Lipschitz with a common Lipschitz constant, which is independent of the control, and hence, their infimum, i.e., the value function, is Lipschitz with the same Lipschitz constant. The necessary conditions of HDP are then established in the form of the HJB equation and the corresponding boundary conditions.

In Section VI, the main result is given describing the relationship of the MP and DP for classical and hybrid systems. The proof is different in the approach from the classical arguments discussed earlier, and in particular, the sequence of proof steps appears in a different order. To be specific, the optimality condition, i.e., the Hamiltonian minimization property (ii) discussed earlier, appears in the last step in order to emphasize the independence of the dynamics of the cost gradient process from the optimality of the control input. Consequently, assumption (i) is used differently here from the classical proof methods, and in particular, the optimality of the transitions back to the reference trajectory is relaxed to the existence of (not necessarily optimal) neighboring trajectories. After the derivation of the dynamics and boundary conditions for the cost gradient, or sensitivity, corresponding to an arbitrary control input, it is shown that an optimal control leads to the same dynamics and boundary conditions for the optimal cost gradient process as those for the adjoint process. Thus, by the existence and uniqueness of the solutions to the governing ordinary differential equations (ODE), it is concluded that the optimal cost gradient, i.e., the gradient of the value function generated by the HJB, is equal to the adjoint process in the corresponding HMP formulation.

Illustrative examples are provided in Section VII.

II. Hybrid Systems

Definition 1: A *hybrid system (structure)* $\mathbb{H}$ is a septuple

$$\mathbb{H} = \{H, I, \Gamma, A, F, \Xi, \mathscr{M}\} \tag{1}$$

where the symbols in the expression and their governing assumptions are defined as below.

A0: $H := Q \times M$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$, where

$Q = \{1, 2, ..., |Q|\} \equiv \{q_1, q_2, ..., q_{|Q|}\}, |Q| < \infty$, is a finite set of *discrete states (components)*, and

$M = \{\mathbb{R}^{n_q}\}_{q\in Q}$ is a family of finite dimensional continuous valued state spaces, where $n_q \le n < \infty$ for all $q \in Q$.

$I := \Sigma \times U$ is the set of system input values, where

Σ with $|\Sigma| < \infty$ is the set of discrete-state transition and continuous state jump events extended with the identity element, and

$U = \{U_q\}_{q\in Q}$ is the set of *admissible input control values*, where each $U_q \subset \mathbb{R}^{m_q}$ is a compact set in $\mathbb{R}^{m_q}$.

The set of admissible (continuous) control inputs $\mathscr{U}(U) := L_\infty([t_0, T_*), U)$ is defined to be the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*), T_* < \infty$. The boundedness property necessarily holds since admissible input functions take values in the compact set U.

$\Gamma : H \times \Sigma \to H$ is a time-independent (partially defined) *discrete-state transition map*.

$\Xi : H \times \Sigma \to H$ is a time-independent (partially defined) *continuous-state jump transition map*. All $\xi_\sigma \in \Xi$, $\xi_\sigma : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$, $p \in A(q, \sigma)$ are assumed to be continuously differentiable in the continuous state $x \in \mathbb{R}^{n_q}$.

$A : Q \times \Sigma \to Q$ denotes both a deterministic finite automaton and the automaton's associated transition function on the state space Q and event set Σ, such that for a discrete state $q \in Q$, only the discrete controlled and uncontrolled transitions into the q-dependent subset $\{A(q, \sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : H \times \Sigma \to H|_Q$. In other words, Γ can only make a discrete-state transition in a hybrid state (q, x) if the automaton A can make the corresponding transition in q.

F is an indexed collection of *vector fields* $\{f_q\}_{q\in Q}$ such that $f_q \in C^{k_{f_q}}(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}^{n_q})$, $k_{f_q} \ge 1$, satisfies a joint uniform Lipschitz condition, i.e., there exists $L_f < \infty$ such that $\|f_q(x_1, u_1) - f_q(x_2, u_2)\| \le L_f(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $x, x_1, x_2 \in \mathbb{R}^{n_q}$, $u, u_1, u_2 \in U_q$, $q \in Q$.

$\mathscr{M} = \{m_\alpha : \alpha \in Q \times Q,\}$ denotes a collection of *switching manifolds* such that, for any ordered pair $\alpha \equiv (\alpha_1, \alpha_2) = (p, q)$, m_α is a smooth, i.e., C^∞, codimension 1 submanifold of $\mathbb{R}^{n_{\alpha_1}}$, described locally by $m_\alpha = \{x : m_\alpha(x) = 0\}$. It is assumed that $m_\alpha \cap m_\beta = \emptyset$, whenever $\alpha_1 = \beta_1$, but $\alpha_2 \ne \beta_2$, for all $\alpha, \beta \in Q \times Q$. ∎

We note that the case where m_α is identified with its reverse ordered version $m_{\bar\alpha}$ giving $m_\alpha = m_{\bar\alpha}$ is not ruled out by this definition, even in the nontrivial case $m_{p,p}$, where $\alpha_1 = \alpha_2 = p$. The former case corresponds to the common situation where the switching of vector fields at the passage of the continuous trajectory in one direction through a switching manifold is reversed if a reverse passage is performed by the continuous trajectory, while the latter case corresponds to the standard example of the bouncing ball.

Switching manifolds will function in such a way that whenever a trajectory governed by the controlled vector field meets the switching manifold transversally, there is an autonomous switching to another controlled vector field or there is a jump transition in the continuous state component, or both. A transversal arrival on a switching manifold $m_{q,r}$ at state $x_q \in m_{q,r} = \{x \in \mathbb{R}^{n_q} : m_{q,r}(x) = 0\}$ occurs whenever

$$\nabla m_{q,r}\left(x_q\right)^T f_q\left(x_q, u_q\right) \neq 0 \tag{2}$$

for $u_q \in U_q$, and $q, r \in Q$. We further assume the following:

A1: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0,q_j}(x_0) \neq 0$, for all $q_j \in Q$. ■

Definition 2: A *hybrid input process* is a pair $I_L \equiv I_L^{[t_0,t_f)} := (S_L, u)$ defined on a half open interval $[t_0, t_f)$, $t_f < \infty$, where $u \in \mathscr{U}$ and $S_L = ((t_0,\sigma_0),(t_1,\sigma_1),\ldots,(t_L,\sigma_L))$, $L < \infty$, is a finite *hybrid sequence of switching events* consisting of a strictly increasing sequence of times $\tau_L := \{t_0, t_1, t_2, \ldots, t_L\}$ and a *discrete event sequence* σ with $\sigma_0 = id$ and $\sigma_i \in \Sigma$, $i \in \{1, 2, \ldots, L\}$. ■

Definition 3: A *hybrid state process* (or *trajectory*) is a triple (τ_L, q, x) consisting of the sequence of switching times $\tau_L = \{t_0, t_1, \ldots, t_L\}$, $L < \infty$, the associated sequence of discrete states $q = \{q_0, q_1, \ldots, q_L\}$, and the sequence $x(\cdot) = \{x_{q_0}(\cdot), x_{q_1}(\cdot), \ldots, x_{q_L}(\cdot)\}$ of piecewise differentiable functions $x_{q_i}(\cdot) : [t_i, t_{i+1}) \to \mathbb{R}^{n_{q_i}}$. ■

Definition 4: The *input-state trajectory* for the hybrid system $\mathbb{H}$ satisfying A0 and A1 is a hybrid input $I_L = (S_L, u)$ together with its corresponding hybrid state trajectory (τ_L, q, x) defined over $[t_0, t_f)$, $t_f < \infty$, such that it satisfies the following:

i) *Continuous state dynamics:* The continuous state component $x(\cdot) = \{x_{q_0}(\cdot), x_{q_1}(\cdot), \ldots, x_{q_L}(\cdot)\}$ is a piecewise continuous function, which is almost everywhere differentiable and on each time segment specified by τ_L satisfies the dynamics equation

$$\dot{x}_{q_i}(t) = f_{q_i}\left(x_{q_i}(t), u_{q_i}(t)\right), \quad a.e.\ t \in [t_i, t_{i+1}) \tag{3}$$

with the initial conditions

$$x_{q_0}(t_0) = x_0, \tag{4}$$

$$x_{q_i}(t_i) = \xi_{\sigma_i}\left(x_{q_{i-1}}(t_i-)\right) := \xi_{\sigma_i}\left(\lim_{t\uparrow t_i} x_{q_{i-1}}(t)\right) \tag{5}$$

for $(t_i, \sigma_i) \in S_L$. In other words, $x(\cdot) = \{x_{q_0}(\cdot), x_{q_1}(\cdot), \ldots, x_{q_L}(\cdot)\}$ is a piecewise continuous function, which is almost everywhere differentiable and is such that each $x_{q_i}(\cdot)$ satisfies

$$x_{q_i}(t) = x_{q_i}(t_i) + \int_{t_i}^{t} f_{q_i}\left(x_{q_i}(s), u_{q_i}(s)\right) ds \tag{6}$$

for $t \in [t_i, t_{i+1})$.

ii) *Autonomous discrete transition dynamics:* An autonomous (uncontrolled) discrete-state transition from q_{i-1} to q_i together with a continuous state jump ξ_{σ_i} occurs at the *autonomous switching time* t_i if $x_{q_{i-1}}(t_i-) := \lim_{t\uparrow t_i} x_{q_{i-1}}(t)$ satisfies a switching manifold condition of the form

$$m_{q_{i-1}q_i}\left(x_{q_{i-1}}(t_i-)\right) = 0 \tag{7}$$

for $q_{i-1}, q_i \in Q$, where $m_{q_{i-1}q_i}(x) = 0$ defines a (q_{i-1}, q_i) switching manifold and it is not the case that either (i) $x_{q_{i-1}}(t_i-) \in \partial m_{q_{i-1}q_i}$ or (ii) $f_{q_{i-1}}(x_{q_{i-1}}(t_i-), u_{q_{i-1}}(t_i-)) \perp \nabla m_{q_{i-1}q_i}(x_{q_{i-1}}(t_i-))$, i.e., t_i is not a manifold termination instant (see [52]). With Assumptions A0 and A1 in force, such a transition is well defined and labels the event $\sigma_{q_{i-1}q_i} \in \Sigma$, that corresponds to the hybrid state transition

$$\begin{aligned} h(t_i) &\equiv \left(q_i, x_{q_i}(t_i)\right) \\ &= \left(\Gamma(q_{i-1}, \sigma_{q_{i-1}q_i}), \xi_{\sigma_{q_{i-1}q_i}}(x_{q_{i-1}}(t_i-))\right). \end{aligned} \tag{8}$$

iii) *Controlled discrete transition dynamics:* A controlled discrete-state transition together with a controlled continuous state jump ξ_σ occurs at the *controlled discrete event time* t_i if t_i is not an autonomous discrete event time and if there exists a controlled discrete input event $\sigma_{q_{i-1}q_i} \in \Sigma$ for which

$$\begin{aligned} h(t_i) &\equiv \left(q_i, x_{q_i}(t_i)\right) \\ &= \left(\Gamma(q_{i-1}, \sigma_{q_{i-1}q_i}), \xi_{\sigma_{q_{i-1}q_i}}(x_{q_{i-1}}(t_i-))\right) \end{aligned} \tag{9}$$

with $(t_i, \sigma_{q_{i-1}q_i}) \in S_L$ and $q_i \in A(q_{i-1})$. ■

Theorem 2.1 (Existence and uniqueness of solution trajectories for hybrid systems [52]): *A hybrid system $\mathbb{H}$ with an initial hybrid state (q_0, x_0) satisfying Assumptions A0 and A1 possesses a unique hybrid input-state trajectory on $[t_0, T_{**})$, where T_{**} is the least of*

i) *$T_* \leq \infty$, where $[t_0, T_*)$ is the temporal domain of the definition of the hybrid system;*

ii) *a manifold termination instant T_* of the trajectory $h(t) = h(t, (q_0, x_0), (S_L, u))$, $t \geq t_0$, at which either $x(T_*-) \in \partial m_{q(T_*-)q(T_*)}$ or $f_{q(T_*-)}(x(T_*-), u(T_*-)) \perp \nabla m_{q(T_*-)q(T_*)}(x(T_*-))$.* ■

We note that Zeno times, i.e., accumulation points of discrete transition times, are ruled out by Definitions 2–4.

III. Hybrid Optimal Control Problems

A2: Let $\{l_q\}_{q\in Q}, l_q \in C^{n_l}(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}_+), n_l \geq 1$, be a family of *running cost* functions with $n_l = 2$ unless otherwise stated; $\{c_{\sigma_j}\}_{\sigma_j\in\Sigma} \in C^{n_c}(\mathbb{R}^{n_{q_{j-1}}} \times \Sigma \to \mathbb{R}_+), n_c \geq 1$, be a family of *switching cost* functions; and $g \in C^{n_g}(\mathbb{R}^{n_{q_f}} \to \mathbb{R}_+), n_g \geq 1$, be a *terminal cost* function satisfying the following assumptions.

i) There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $|l_q(x,u)| \leq K_l(1 + \|x\|^{\gamma_l})$ and $|l_q(x_1,u_1) - l_q(x_2,u_2)| \leq K_l(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $x \in \mathbb{R}^{n_q}, u \in U_q, q \in Q$.

ii) There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c_{\sigma_j}(x)| \leq K_c(1 + \|x\|^{\gamma_c}), x \in \mathbb{R}^{n_{q_{j-1}}}, \sigma_j \in \Sigma$.

iii) There exists $K_g < \infty$ and $1 \leq \gamma_g < \infty$ such that $|g(x)| \leq K_g(1 + \|x\|^{\gamma_g}), x \in \mathbb{R}^{n_{q_f}}$. ■

Consider the initial time t_0, final time $t_f < \infty$, and initial hybrid state $h_0 = (q_0, x_0)$. With the number of switchings L held fixed, the set of all hybrid input trajectories in Definition 2 with exactly L switchings is denoted by $\boldsymbol{I_L}$, and for all $I_L := (S_L, u) \in \boldsymbol{I_L}$, the hybrid switching sequences take the form $S_L = \{(t_0, id), (t_1, \sigma_{q_0 q_1}), \ldots, (t_L, \sigma_{q_{L-1} q_L})\} \equiv \{(t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L)\}$ and the corresponding continuous control inputs are of the form $u \in \mathscr{U} = \bigcup_{i=0}^{L} L_\infty([t_i, t_{i+1}), U_{q_i})$, where $t_{L+1} = t_f$.

Let I_L be a hybrid input trajectory that by Theorem 2.1 results in a unique hybrid state process. Then, hybrid performance functions for the corresponding hybrid input-state trajectory are defined as

$$J(t_0, t_f, h_0, L; I_L) := \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u_{q_i}(s))\, ds + \sum_{j=1}^{L} c_{\sigma_j}\left(t_j, x_{q_{j-1}}(t_j-)\right) + g(x_{q_L}(t_f)). \quad (10)$$

Definition 5: The *Bolza hybrid optimal control problem* (BHOCP) is defined as the infimization of the hybrid cost (10) over the family of hybrid input trajectories $\boldsymbol{I_L}$, i.e.,

$$J^o(t_0, t_f, h_0, L) = \inf_{I_L \in \boldsymbol{I_L}} J(t_0, t_f, h_0, L; I_L). \quad (11)$$

■

Definition 6: The *Mayer hybrid optimal control problem* (MHOCP) is defined as a special case of the BHOCP where $l_q(x_q, u_q) = 0$ for all $q \in Q$ and $c_{\sigma_j}(t_j, x_{q_{j-1}}(t_j-)) = 0$ for all $\sigma_j \in \Sigma$, $1 \le j \le L$. ■

Remark 3.1: The relationship between the BHOCP and the MHOCP: In general, a BHOCP can be converted into an MHOCP with the introduction of an auxiliary state component z and the extension of the continuous valued state to

$$\hat{x}_q := \begin{bmatrix} z_q \\ x_q \end{bmatrix}. \quad (12)$$

With the definition of the augmented vector fields

$$\dot{\hat{x}}_q = \hat{f}_q(\hat{x}_q, u_q) := \begin{bmatrix} l_q(x_q, u_q) \\ f_q(x_q, u_q) \end{bmatrix} \quad (13)$$

subject to the initial condition

$$\hat{h}_0 = (q_0, \hat{x}_{q_0}(t_0)) = \left(q_0, \begin{bmatrix} 0 \\ x_0 \end{bmatrix}\right) \quad (14)$$

and with the switching boundary conditions governed by the extended jump function defined as

$$\hat{x}_{q_j}(t_j) = \hat{\xi}_{\sigma_j}\left(\hat{x}_{q_{j-1}}(t_j-)\right) := \begin{bmatrix} z_{q_{j-1}}(t_j-) + c_{\sigma_j}\left(x_{q_{j-1}}(t_j-)\right) \\ \xi_{\sigma_j}\left(x_{q_{j-1}}(t_j-)\right) \end{bmatrix} \quad (15)$$

the cost (10) of the BHOCP turns into the Mayer form with

$$J\left(t_0, t_f, \hat{h}_0, L; I_L\right) := \hat{g}(\hat{x}_{q_L}(t_f)) \quad (16)$$

where $\hat{g}(\hat{x}_{q_L}(t_f)) = z_{q_L}(t_f) + g(x_{q_L}(t_f))$. ■

IV. Hybrid Minimum Principle

Theorem 4.1 ([53]): *Consider the hybrid system $\mathbb{H}$ subject to Assumptions A0–A2 and the HOCP (11) for the hybrid performance function (10). Define the family of system Hamiltonians by*

$$H_q(x_q, \lambda_q, u_q) = \lambda_q^T f_q(x_q, u_q) + l_q(x_q, u_q) \quad (17)$$

$x_q, \lambda_q \in \mathbb{R}^{n_q}$, $u_q \in U_q$, $q \in Q$. Then, for a given switching sequence $\{q_i\}_{i=0}^{L}$ and along the corresponding optimal trajectory x^o, there exists an adjoint process λ^o such that

$$\dot{x}_q^o = \frac{\partial H_q}{\partial \lambda_q}(x_q^o, \lambda_q^o, u_q^o) \quad (18)$$

$$\dot{\lambda}_q^o = -\frac{\partial H_q}{\partial x_q}(x_q^o, \lambda_q^o, u_q^o) \quad (19)$$

almost everywhere $t \in [t_0, t_f]$ with

$$x_{q_0}^o(t_0) = x_0 \quad (20)$$

$$x_{q_j}^o(t_j) = \xi_{\sigma_j}\left(x_{q_{j-1}}^o(t_j-)\right) \quad (21)$$

$$\lambda_{q_L}^o(t_f) = \nabla g\left(x_{q_L}^o(t_f)\right) \quad (22)$$

$$\lambda_{q_{j-1}}^o(t_j-) \equiv \lambda_{q_{j-1}}^o(t_j) = \nabla \xi_{\sigma_j}{}^T \lambda_{q_j}^o(t_j+) + \nabla c_{\sigma_j} + p\nabla m \quad (23)$$

where $p \in \mathbb{R}$, when t_j indicates the time of an autonomous switching, and $p = 0$ when t_j indicates the time of a controlled switching. Moreover

$$H_q(x^o, \lambda^o, u^o) \le H_q(x^o, \lambda^o, u) \quad (24)$$

for all $u \in U_{q^o}$, that is to say the Hamiltonian is minimized with respect to the control input, and at a switching time t_j, the Hamiltonian satisfies

$$H_{q_{j-1}}(x^o, \lambda^o, u^o)\big|_{t_j-} \equiv H_{q_{j-1}}(t_j) = H_{q_j}(t_j) \equiv H_{q_j}(x^o, \lambda^o, u^o)\big|_{t_j+}. \quad (25)$$

■

We note that the gradient of the state transition jump map $\nabla \xi_{\sigma_j} = \frac{\partial \xi_{\sigma_j}(x_{q_{j-1}})}{\partial x_{q_{j-1}}} \equiv \frac{\partial x_{q_j}}{\partial x_{q_{j-1}}}$ is not necessarily square due to the possibility of changes in the state dimension, but the boundary conditions (23) are well defined for hybrid optimal control problems satisfying A0–A2.

Remark 4.2: For an HOCP represented in the Bolza form (as in Definition 5) and the corresponding Mayer representation (as in Definition 6 and through Remark 3.1), the corresponding HMP results are given by (see also [20])

$$\hat{\lambda}_q^T \hat{f}_q(\hat{x}_q, u_q) \equiv \hat{H}_q\left(\hat{x}, \hat{\lambda}, u\right) = H_q(x, \lambda, u) \equiv \lambda_q^T f_q(x_q, u_q) + l_q(x_q, u_q), \quad (26)$$

$$\hat{\lambda}_q(t) = \begin{bmatrix} 1 \\ \lambda_q(t) \end{bmatrix}. \quad (27)$$

■

V. Hybrid Dynamic Programming

Definition 7: Consider the hybrid system (1) and the class of HOCP (11) with the hybrid cost (10). At an arbitrary instant $t \in [t_0, t_f]$, an initial hybrid state $h = (q, x) \in Q \times \mathbb{R}^{n_q}$, and a specified number of remaining switchings $L - j + 1$ corresponding to $t \in (t_{j-1}, t_j]$, the *cost to go* subject to the hybrid input process $I_{L-j+1} \equiv I_{L-j+1}^{[t,t_f]}$ is given by

$$\begin{aligned} &J(t, t_f, q, x, L-j+1; I_{L-j+1}) \\ &= \int_t^{t_j} l_q(x_q, u_q)\, ds + \sum_{i=j}^{L} c_{\sigma_{q_{i-1}q_i}}(t_i, x_{q_{i-1}}(t_i-)) \\ &+ \sum_{i=j}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}, u_{q_i})\, ds + g(x_{q_L}(t_f)). \end{aligned} \tag{28}$$

The *value function* is defined as the optimal cost to go over the corresponding family of hybrid control inputs, i.e.,

$$V(t, q, x, L-j+1) := \inf_{I_{L-j+1}} J(t, t_f, q, x, L-j+1; I_{L-j+1}). \tag{29}$$

■

Theorem 5.1: *With Assumptions A0–A2 in force, the value function (29) is Lipschitz in x uniformly in t for all $t \in \bigcup_{i=0}^{L}(t_i, t_{i+1})$, i.e., for $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$ and for all $t \in (t_i, t_{i+1})$, $x \equiv x_t \in B_r$, there exist a neighborhood $N_{r_x}(x_t)$ and a constant $0 < K < \infty$ such that*

$$\begin{aligned} &|V(t, q, x_t, L-j+1) - V(s, q, x_s, L-j+1)| \\ &< K\left(\|x_t - x_s\|^2 + |s-t|^2\right)^{\frac{1}{2}} \end{aligned} \tag{30}$$

for $s \in (t_i, t_{i+1})$ and $x_s \in N_{r_x}(x_t)$. ■

Proof: See the Appendix.

Definition 8: Let $M_{(i)}$ denote the set of all $(t, x) \in \bigcup_{i=0}^{L} \mathbb{R} \times \mathbb{R}^{n_{q_i}}$, for which the ith derivatives of V exist and are continuous. ■

Note that from Theorem 5.1, it is concluded that $M_{(0)} \supseteq \bigcup_{i=0}^{L}(t_i, t_{i+1}) \times \mathbb{R}^{n_{q_i}}$, i.e., the value function is at most discontinuous at the switching instants with nonzero switching costs and nonidentity jump maps.

Definition 9: A hybrid feedback input $I_{L-j+1}^{[t,t_f]}(t, q, x) = (S_{L-j+1}, u_{q(\tau)}(\tau, x))$, $\tau \in [t, t_f]$ is said to have an *admissible set of discontinuities*, if for each $q \in Q$, the discontinuities of the continuous valued feedback control $u_q(t, x)$ and the discrete valued feedback input $\sigma(t, q, x)$ are located on lower dimensional manifolds in the time and state space $\mathbb{R} \times \mathbb{R}^{n_q}$. ■

Remark 5.2: By A0, an autonomous discrete-valued control input σ necessarily satisfies the lower dimensional manifold switching set condition of Definition 9, where the sets constitute C^∞ submanifolds.

Remark 5.3: For classical (i.e., nonhybrid) systems, a more detailed definition of a feedback control law with an admissible set of discontinuities can be found in [8, pp. 90–97]. The necessary conditions for the Lipschitz continuity of the optimal feedback control are discussed in [54] and [55], and sufficient conditions for continuity with respect to initial conditions are given in [56].

Corollary 5.4: From Theorem 5.1 and Rademacher's theorem (see, e.g., [8], [57], and [58]), the Lipschitz property implies differentiability of the value function almost everywhere. Furthermore, if there exists an optimal input with an admissible set of discontinuities, $M_{(1)}$ is open dense in $\bigcup_{i=0}^{L}[t_i, t_{i+1}] \times \mathbb{R}^{n_{q_i}}$. ■

Theorem 5.5 ([53]): *Consider the hybrid system $\mathbb{H}$ and the HOCP (11) together with the Assumptions A0–A2 as above. Then, for all $(t, x_{q_i}) \in M_{(1)}$, $q_i \in Q$, the HJB equation holds, i.e.,*

$$-\frac{\partial V}{\partial t} = \inf_{u_{q_i}} \{l_{q_i}(x_{q_i}, u_{q_i}) + \langle \nabla V, f_{q_i}(x_{q_i}, u_{q_i})\rangle\} \tag{31}$$

a.e. $t \in (t_i, t_{i+1})$, $0 \le i \le L$, subject to the terminal condition

$$V(t_f, q_L, x, 0) = g(x_{q_L}) \tag{32}$$

and at the switching times $t_j \in \tau_L = \{t_1, \cdots, t_L\}$, subject to the boundary conditions

$$\begin{aligned} &V(t_j, q, x, L-j+1) \\ &= \min_{\sigma_j \in \Sigma_j} \left\{V\left(t_j, \Gamma(q, \sigma_j), \xi_{\sigma_j}(x), \; L-j\right) + c_{\sigma_j}(x)\right\} \end{aligned} \tag{33}$$

$$\begin{aligned} &l_q(x, u^o(t_j-, x)) + \langle \nabla V, f_q(x, u^o(t_j-, x))\rangle \\ &\equiv \frac{-\partial}{\partial t} V(t_j-, q, x, L-j+1) \\ &= \frac{-\partial}{\partial t} V\left(t_j, \Gamma(q, \sigma_j), \xi_{\sigma_j}(x), L-j\right) \\ &\equiv l_{\Gamma(q,\sigma_j)}\left(\xi_{\sigma_j}^{(x)}, u^o\left(t_j, \xi_{\sigma_j}^{(x)}\right)\right) \\ &\quad + \left\langle \nabla V, f_{\Gamma(q,\sigma_j)}\left(\xi_{\sigma_j}^{(x)}, u^o\left(t_j, \xi_{\sigma_j}^{(x)}\right)\right)\right\rangle \end{aligned} \tag{34}$$

where if $(t_j, x) \in \mathbb{R} \times \mathbb{R}^{n_{q_{j-1}}}$ belong to a controlled switching set, then $\Sigma_j = \Sigma$ subject to the automaton constraint that $\Gamma(q, \sigma_j)$ is defined, and in the case of an autonomous switching, the set Σ_j is reduced to a subset of discrete inputs, which are consistent with the switching manifold condition $m_{q,\Gamma(q,\sigma_j)}(x_q) = 0$. In the above equation, the notation $u^o(t, x)$ indicates the (continuous valued) optimal input corresponding to x at the instant t. ■

VI. Relationship Between the HMP and HDP

Theorem 6.1 (Evolution of the cost sensitivity along a general trajectory): *Consider the hybrid system $\mathbb{H}$ together with Assumptions A0–A2 and the hybrid cost to go (28). Then, for a given hybrid feedback control $I_{L-j+1}^{[t,t_f]}(t, q, x)$ with an admissible set of discontinuities, the sensitivity function $\nabla J \equiv \frac{\partial}{\partial x} J(t, t_f, q, x, L-j+1; I_{L-j+1}), 1 \le j < L+1$, satisfies*

$$\frac{d}{dt}\nabla J = -\left(\left[\frac{\partial f_q(x, u)}{\partial x}\right]^T \nabla J + \frac{\partial l_q(x, u)}{\partial x}\right) \tag{35}$$

subject to the terminal condition:

$$\nabla J(t_f, q_L, x, 0; I_0) = \nabla g(x) \tag{36}$$

and the boundary conditions:

$$\begin{aligned}&\nabla J\left(t_j-, q_{j-1}, x, L-j+1; I_{L-j+1}\right)\\&\equiv \nabla J\left(t_j, q_{j-1}, x, L-j+1; I_{L-j+1}\right)\\&=\nabla \xi_{\sigma_j}\big|_x^T \nabla J\big(t_j+, q_j, \xi_{\sigma_j}(x), L-j; I_{L-j}\big)+p\,\nabla m|_x+\nabla c|_x\end{aligned} \tag{37}$$

with $p=0$ *when* $(t_j, x) \in \mathbb{R}\times\mathbb{R}^{n_{q_{j-1}}}$ *belong to a controlled switching set, and*

$$p=\frac{\left[\nabla J\left(t_j+, q_j, \xi_{\sigma_j}(x), L-j; I_{L-j}\right)\right]^T f_{q_j,\xi}^{\xi, q_{j-1}}+l_{q_j,\xi}^{q_{j-1}}}{\nabla m^T f_{q_{j-1}}\left(x, u\left(t_j-\right)\right)} \tag{38}$$

when $(t_j, x) \in \mathbb{R}\times\mathbb{R}^{n_{q_{j-1}}}$ *belong to an autonomous switching set, and where in the above equation* $f_{q_j,\xi}^{\xi,q_{j-1}} := f_{q_j}(\xi_{\sigma_j}(x_{q_{j-1}}(t_j^-)), u_{q_j}(t_j)) - \nabla\xi f_{q_{j-1}}(x_{q_{j-1}}(t_j^-), u_{q_{j-1}}(t_j^-))$ *and* $l_{q_j,\xi}^{q_{j-1}} := l_{q_j}(\xi_{\sigma_j}(x_{q_{j-1}}), u_{q_j}(t_j)) - l_{q_{j-1}}(x_{q_{j-1}}, u_{q_{j-1}}(t_j^-))$. ■

Proof: We first prove that (35) holds for $t\in(t_L, t_{L+1}]\equiv(t_L, t_f]$ with the terminal condition (36). Then, by assuming that (35) holds for $t\in(t_j, t_{j+1}]$, $j\le L$, we show that it also holds for $t\in(t_{j-1}, t_j]$ with the boundary condition (37), with $p=0$ when t_j indicates the time of a controlled switching, and with p given by (38) when $t_j\in\tau_L$ indicates the time of an autonomous switching. Hence, by mathematical induction, the relation is proved for all $t\in[t_0, t_f]$.

i) *No switching ahead:* First, consider a Lebesgue time $t\in[t_L, t_{L+1}]\equiv[t_L, t_f]$ and the hybrid trajectory passing through (q_L, x), and consider the cost to go (28) for I_0, which is

$$J\left(t, q_L, x, 0; I_0\right)=\int_t^{t_f} l_{q_L}\left(x_s, u_s\right) ds+g\left(x_f\right). \tag{39}$$

Since by Definition 9 the discontinuities in x of $I_0^{[t,t_f]}\equiv u^{[t,t_f]}$ lie on lower dimensional sets which are closed in the induced topology of the space, the partial derivative of J with respect to x exists in an open neighborhood of (t,x) and is derived as

$$\begin{aligned}\frac{\partial J\left(t, q_L, x, 0; I_0\right)}{\partial x}&=\frac{\partial}{\partial x}\int_t^{t_f} l_{q_L}\left(x_{q_L}^{(s)}, u_{q_L}^{(s)}\right) ds+\frac{\partial}{\partial x} g\left(x_{q_L}^{(t_f)}\right)\\&=\int_t^{t_f}\frac{\partial}{\partial x} l_{q_L}\left(x_s, u_s\right) ds+\frac{\partial}{\partial x} g\left(x_f\right)\end{aligned} \tag{40}$$

which is equivalent to

$$\begin{aligned}\frac{\partial J\left(t, q_L, x, 0; I_0\right)}{\partial x}=&\int_t^{t_f}\left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_L}\left(x_s, u_s\right)}{\partial x_s} ds\\&+\left[\frac{\partial x_f}{\partial x}\right]^T \frac{\partial g(x_f)}{\partial x_f}.\end{aligned} \tag{41}$$

Taking $t=t_f$, the terminal condition for $\frac{\partial J}{\partial x}$ is seen to be determined by

$$\frac{\partial J\left(t_f, q_L, x, 0; I_0\right)}{\partial x}=\nabla_{x_f} g\left(x_f\right)\equiv\nabla g(x) \tag{42}$$

because $x_f=x$ when $t=t_f$. Hence, (36) is proved. With the notation $x_s=\phi_{q_L}(s,t,x)$ and with the smoothness provided by Assumptions A0–A2 for the given control input with an admissible set of discontinuities, we have

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial}{\partial x} x_s\right)&=\frac{d}{ds}\left(\frac{\partial}{\partial x}\phi_{q_L}(s,t,x)\right)=\frac{\partial}{\partial x}\left(\frac{d}{ds}\phi_{q_L}(s,t,x)\right)\\&=\frac{\partial}{\partial x}\left(f_{q_L}\left(\phi_{q_L}(s,t,x), u\right)\right)\end{aligned} \tag{43}$$

from which we obtain

$$\frac{d}{ds}\left(\frac{\partial x_s}{\partial x}\right)=\left[\frac{\partial f_{q_L}}{\partial x_s}\right]^T \frac{\partial\phi_{q_L}(s,t,x)}{\partial x} \tag{44}$$

with $\frac{\partial\phi_{q_L}(t,t,x)}{\partial x}=I_{n_{q_L}\times n_{q_L}}$, since $\phi_{q_L}(t,t,x)=x$. Let $\Phi_{s,t}^{q_L}\in\mathbb{R}^{n_{q_L}\times n_{q_L}}$ denote the solution of

$$\dot{\Phi}_{s,t}^{q_L}=\nabla_{x_s} f_{q_L}\left(x_s, u_s\right)^T\Phi_{s,t}^{q_L}\equiv\left[\frac{\partial f_{q_L}\left(x_s, u_s\right)}{\partial x_s}\right]^T\Phi_{s,t}^{q_L} \tag{45}$$

with $\Phi_{t,t}^{q_L}=I_{n_{q_L}\times n_{q_L}}$. By the uniqueness of the solutions to (44) and (45):

$$\frac{\partial}{\partial x}\phi_{q_L}(s,t,x)=\Phi_{s,t}^{q_L}, \tag{46}$$

for all $x\in\mathbb{R}^{n_{q_L}}$. Also by the semigroup property:

$$x=\phi_{q_L}\left(s,t,x_t\right)=\phi_{q_L}\left(s,t,\phi_{q_L}(t,s,x)\right). \tag{47}$$

Hence, by taking the derivative with respect to x, we have

$$I_{n_{q_L}\times n_{q_L}}=\left.\frac{\partial\phi_{q_L}(s,t,z)}{\partial z}\right|_{z=\phi(t,s,x_s)}\frac{\partial\phi_{q_L}(t,s,x)}{\partial x} \tag{48}$$

which by (46) is equivalent to

$$I_{n_{q_L}\times n_{q_L}}=\Phi_{s,t}^{q_L}\Phi_{t,s}^{q_L}. \tag{49}$$

For all $r,s,t\in(t_L,t_f]$, it is the case that

$$\frac{d}{ds}\Phi_{s,r}^{q_L}=\left[\frac{\partial f_{q_L}\left(x_s, u_s\right)}{\partial x_s}\right]^T\Phi_{s,r}^{q_L},\quad \Phi_{r,r}^{q_L}=I_{n_{q_L}\times n_{q_L}} \tag{50}$$

$$\begin{aligned}\frac{d}{ds}\left(\Phi_{s,t}^{q_L}\Phi_{t,r}^{q_L}\right)&=\left(\left[\frac{\partial f_{q_L}\left(x_s, u_s\right)}{\partial x_s}\right]^T\Phi_{s,t}^{q_L}\right)\Phi_{t,r}^{q_L}\\&=\left[\frac{\partial f_{q_L}\left(x_s, u_s\right)}{\partial x_s}\right]^T\left(\Phi_{s,t}^{q_L}\Phi_{t,r}^{q_L}\right)\end{aligned} \tag{51}$$

where for (51) at $s=r$, the condition $\Phi_{r,t}^{q_L}\Phi_{t,r}^{q_L}=I_{n_{q_L}\times n_{q_L}}$ holds. Hence, from the uniqueness of the solution to the ODEs (50) and (51), we obtain $\Phi_{s,t}^{q_L}\Phi_{t,r}^{q_L}=\Phi_{s,r}^{q_L}$. Furthermore, (49) gives

$$0=\frac{d\Phi_{s,t}^{q_L}}{dt}\Phi_{t,s}^{q_L}+\Phi_{s,t}^{q_L}\frac{d\Phi_{t,s}^{q_L}}{dt} \tag{52}$$

and hence

$$\begin{aligned}\frac{d\Phi_{s,t}^{q_L}}{dt} &= -\Phi_{s,t}^{q_L}\frac{d\Phi_{t,s}^{q_L}}{dt}\left[\Phi_{t,s}^{q_L}\right]^{-1}\\ &= -\Phi_{s,t}^{q_L}\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T \Phi_{t,s}^{q_L}\left[\Phi_{t,s}^{q_L}\right]^{-1}\\ &= -\Phi_{s,t}^{q_L}\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T. \end{aligned}\tag{53}$$

Differentiating (41) with respect to t along a trajectory (q_L, x) gives Eqn. (54) as shown at the bottom of this page, where the zero terms arise from

$$\frac{d}{dt}\nabla_z l_{q_L}(z,u_s)\big|_{z=\phi_{q_L}(s,t,x)} = \frac{d}{dt}\nabla_{x_s} l_{q_L}(x_s,u_s) = 0 \tag{55}$$

$$\frac{d}{dt}\nabla_z g(z,u_s)\big|_{z=\phi_{q_L}(t_f,t,x)} = \frac{d}{dt}\nabla_{x_f} g(x_f) = 0. \tag{56}$$

Hence

$$\frac{d}{dt}\frac{\partial J(t,q_L,x,0;I_0)}{\partial x} = -\frac{\partial l_{q_L}(x_t,u_t)}{\partial x_t} - \left[\frac{\partial f_{q_L}^{(x_t,u_t)}}{\partial x_t}\right]^T\left\{\int_t^{t_f}\left[\frac{\partial x_s}{\partial x}\right]^T\frac{\partial l_{q_L}^{(x_s,u_s)}}{\partial x_s}ds + \left[\frac{\partial x_f}{\partial x}\right]^T\frac{\partial g(x_f)}{\partial x_f}\right\} \tag{57}$$

which gives

$$\frac{d}{dt}\frac{\partial J(t,q_L,x,0;I_0)}{\partial x} = -\left[\frac{\partial f_{q_L}(x,u)}{\partial x}\right]^T\frac{\partial J(t,q_L,x,0;I_0)}{\partial x} - \frac{\partial l_{q_L}(x,u)}{\partial x} \tag{58}$$

with

$$\frac{\partial J}{\partial x}\left(t_f,q_L,x,0;I_0^{[t_f,t_f]}\right) = \nabla_{x_f} g(x_f) \equiv \nabla_x g(x). \tag{59}$$

ii) *A controlled switching ahead:* Now assume that (35) holds for $\theta \in (t_j, t_{j+1}]$, $j \le L$ when $t_j \in \tau_L$ indicates a time of a controlled switching. Then, for $t_{j-1} < t \le t_j < \theta \le t_{j+1}$

$$J\left(t,q_{j-1},x,L-j+1;I_{L-j+1}^{[t,t_f]}\right) = \int_t^{t_j} l_{q_{j-1}}^{(x_s,u_s)}ds + c_{\sigma_j}^{(x(t_j-))} + \int_{t_j}^{\theta} l_{q_j}(x_\omega,u_\omega)d\omega + J\left(\theta,q_j,x_\theta,L-j;I_{L-j}^{[\theta,t_f]}\right) \tag{60}$$

where

$$x_\theta = \xi\left(x_t + \int_t^{t_j-} f_{q_{j-1}}(x_s,u_s)\,ds\right) + \int_{t_j}^{\theta} f_{q_j}(x_\omega,u_\omega)\,d\omega. \tag{61}$$

This gives

$$\frac{\partial J(t,q_{j-1},x,L-j+1;I_{L-j+1})}{\partial x} = \frac{\partial}{\partial x}\int_t^{t_j} l_{q_{j-1}}(x_s,u_s)ds + \frac{\partial c_{\sigma_j}^{(x(t_j-))}}{\partial x} + \frac{\partial}{\partial x}\int_{t_j}^{\theta} l_{q_j}^{(x_\omega,u_\omega)}d\omega + \frac{\partial J(\theta,q_j,x_\theta,L-j;I_{L-j})}{\partial x} \tag{62}$$

which is equivalent to

$$\begin{aligned}\frac{\partial J(t,q_{j-1},x,L-j+1;I_{L-j+1})}{\partial x} &= \int_t^{t_j}\left[\frac{\partial x_s}{\partial x}\right]^T\frac{\partial l_{q_{j-1}}^{(x_s,u_s)}}{\partial x_s}ds\\ &+ \left[\frac{\partial x_{t_j-}}{\partial x}\right]^T\frac{\partial c(x_{t_j-})}{\partial x_{t_j-}} + \int_{t_j}^{\theta}\left[\frac{\partial x_\omega}{\partial x}\right]^T\frac{\partial l_{q_j}(x_\omega,u_\omega)}{\partial x_\omega}d\omega\\ &+ \left[\frac{\partial x_\theta}{\partial x}\right]^T\frac{\partial J(\theta,q_j,x_\theta,L-j;I_{L-j})}{\partial x_\theta}\end{aligned} \tag{63}$$

with the differentiation of (61) giving

$$\begin{aligned}\frac{\partial x_\theta}{\partial x} &= \int_{t_j}^{\theta}\frac{\partial f_{q_j}(x_\omega,u_\omega)}{\partial x}d\omega + \frac{\partial\xi\left(x_t + \int_t^{t_j} f_{q_{j-1}}(x_s,u_s)\,ds\right)}{\partial x}\\ &= \int_{t_j}^{\theta}\frac{\partial f_{q_j}(x_\omega,u_\omega)}{\partial x}d\omega + \frac{\partial\xi(x_{t_j-})}{\partial x}\end{aligned} \tag{64}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial J(t,q_L,x,0;I_0)}{\partial x} &= \frac{d}{dt}\int_t^{t_f}\frac{\partial\phi_{q_L}(s,t,x)}{\partial x}^T\frac{\partial l_{q_L}(z,u_s)}{\partial z}\bigg|_{z=\phi_{q_L}(s,t,x)}ds + \frac{d}{dt}\frac{\partial\phi_{q_L}(t_f,t,x)}{\partial x}^T\frac{\partial g}{\partial z}\bigg|_{z=\phi_{q_L}(t_f,t,x)}\\ &= -\left\{\left[\frac{\partial\phi_{q_L}(s,t,x)}{\partial x}\right]^T\frac{\partial l_{q_L}(z,u_s)}{\partial z}\bigg|_{z=\phi_{q_L}(s,t,x)}\right\}_{s=t} + \int_t^{t_f}\frac{d}{dt}\left\{\left[\frac{\partial\phi_{q_L}(s,t,x)}{\partial x}\right]^T\frac{\partial l_{q_L}(z,u_s)}{\partial z}\bigg|_{z=\phi_{q_L}(s,t,x)}\right\}ds\\ &\quad + \frac{d}{dt}\left[\frac{\partial\phi_{q_L}(t_f,t,x)}{\partial x}\right]^T\frac{\partial g}{\partial z}\bigg|_{z=\phi_{q_L}(t_f,t,x)}\\ &= -\left\{I_{n\times n}.\frac{\partial l_{q_L}(x_t,u_t)}{\partial x_t}\right\} + \int_t^{t_f}\left\{-\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T\left[\frac{\partial\phi_{q_L}(s,t,x)}{\partial x}\right]^T\frac{\partial l_{q_L}(z,u_s)}{\partial z}\bigg|_{z=\phi_{q_L}(s,t,x)} + 0\right\}ds\\ &\quad + \left\{-\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T\left[\frac{\partial\phi_{q_L}(t_f,t,x)}{\partial x}\right]^T\frac{\partial g(z)}{\partial z}\bigg|_{z=\phi_{q_L}(t_f,t,x)} + 0\right\}\end{aligned} \tag{54}$$

from which we obtain

$$\frac{\partial x_\theta}{\partial x} = \int_{t_j}^{\theta} \left[\frac{\partial x_\omega}{\partial x}\right]^T \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x_\omega} d\omega + \left[\frac{\partial x_{t_j-}}{\partial x}\right]^T \frac{\partial \xi(x_{t_j-})}{\partial x_{t_j-}}. \tag{65}$$

In particular, for $x = x_t$ as $t \uparrow t_j$ and for x_θ as $\theta \downarrow t_j$, the equation (63) becomes

$$\begin{aligned}\frac{\partial J\left(t_j-, q_{j-1}, x_{t_j-}, L-j+1; I_{L-j+1}\right)}{\partial x_{t_j-}} &= \int_{t_j-}^{t_j} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_{j-1}}(x_s, u_s)}{\partial x_s} ds \\ &+ \left[\frac{\partial x_{t_j-}}{\partial x_{t_j-}}\right]^T \frac{\partial c(x_{t_j-})}{\partial x_{t_j-}} + \int_{t_j}^{t_j+} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_j}(x_s, u_s)}{\partial x_s} ds \\ &+ \left[\frac{\partial x_{t_j+}}{\partial x_{t_j-}}\right]^T \frac{\partial J\left(t_j+, q_j, x_{t_j+}, L-j; I_{L-j}\right)}{\partial x_{t_j+}}\end{aligned} \tag{66}$$

which is equivalent to

$$\frac{\partial J\left(t_j-, q_{j-1}, x_{t_j-}, L-j+1; I_{L-j+1}\right)}{\partial x_{t_j-}} = \frac{\partial c(x_{t_j-})}{\partial x_{t_j-}} + \left[\frac{\partial x_{t_j+}}{\partial x_{t_j-}}\right]^T \frac{\partial J\left(t_j+, q_j, x_{t_j+}, L-j; I_{L-j}\right)}{\partial x_{t_j+}} \tag{67}$$

and also (65) turns into

$$\begin{aligned}\frac{\partial x_{t_j+}}{\partial x_{t_j-}} &= \int_{t_j}^{t_j+} \left[\frac{\partial x_\omega}{\partial x}\right]^T \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x_\omega} d\omega \\ &+ \left[\frac{\partial x_{t_j-}}{\partial x_{t_j-}}\right]^T \frac{\partial \xi(x_{t_j-})}{\partial x_{t_j-}}\end{aligned} \tag{68}$$

which gives

$$\frac{\partial x_{t_j+}}{\partial x_{t_j-}} = \frac{\partial \xi(x_{t_j-})}{\partial x_{t_j-}} = \nabla\xi|_{x_{t_j-}}. \tag{69}$$

Hence

$$\frac{\partial J\left(t_j-, q_{j-1}, x_{t_j-}, L-j+1; I_{L-j+1}^{[t_j, t_f]}\right)}{\partial x_{t_j-}} = \frac{\partial c(x_{t_j-})}{\partial x_{t_j-}} + \nabla\xi|_{x_{t_j-}}^T \frac{\partial J\left(t_j+, q_j, x_{t_j+}, L-j; I_{L-j}^{[t_j+, t_f]}\right)}{\partial x_{t_j+}}. \tag{70}$$

Therefore, (37) is shown to hold with $p = 0$ for the controlled switching case. Writing

$$\begin{aligned}J(t, q_{j-1}, x, 0; I_{L-j+1}) &= \int_t^{t_j} l_{q_{j-1}}(x_s, u_s) ds \\ &+ J(t, q_{j-1}, x(t_j-), L-j+1; I_{L-j+1})\end{aligned} \tag{71}$$

and following a similar procedure as in part (i) of the proof, (35) is derived for $t \in (t_{j-1}, t_j]$.

iii) An autonomous switching ahead: Now assume that (35) holds for all $\theta \in (t_j, t_{j+1}]$, $j \le L$ when $t_j \in \tau_L$ indicates a time of an autonomous switching. Then, taking the derivative of both sides of the equality (60) with respect to x at $t \in (t_{j-1}, t_j]$, with $t_{j-1} < t \le t_j < \theta \le t_{j+1}$, yields

$$\begin{aligned}\frac{\partial J(t, q_{j-1}, x, L-j+1; I_{L-j+1})}{\partial x} &= \frac{\partial}{\partial x}\int_t^{t_j} l_{q_{j-1}}^{(x_s, u_s)} ds + \frac{\partial c(x(t_j-))}{\partial x} \\ &+ \frac{\partial}{\partial x}\int_{t_j}^{\theta} l_{q_j}(x_\omega, u_\omega) d\omega + \frac{\partial J(\theta, q_j, x_\theta, L-j; I_{L-j})}{\partial x}\end{aligned} \tag{72}$$

which gives

$$\begin{aligned}\frac{\partial J(t, q_{j-1}, x, L-j+1; I_{L-j+1})}{\partial x} &= \frac{\partial t_j}{\partial x} l_{q_{j-1}}(x_s, u_s)\big|_{s=t_j-} \\ &+ \int_t^{t_j} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_{j-1}}(x_s, u_s)}{\partial x_s} ds + \left[\frac{\partial x_{t_j-}}{\partial x}\right]^T \frac{\partial c(x_{t_j-})}{\partial x_{t_j-}} \\ &- \frac{\partial t_j}{\partial x} l_{q_j}(x_\omega, u_\omega)\big|_{\omega=t_j} + \int_{t_j}^{\theta} \left[\frac{\partial x_\omega}{\partial x}\right]^T \frac{\partial l_{q_j}(x_\omega, u_\omega)}{\partial x_\omega} d\omega \\ &+ \left[\frac{\partial x_\theta}{\partial x}\right]^T \frac{\partial J(\theta, q_j, x_\theta, L-j; I_{L-j})}{\partial x_\theta}\end{aligned} \tag{73}$$

with the derivative of (61) derived as

$$\begin{aligned}\frac{\partial x_\theta}{\partial x} &= \frac{\partial}{\partial x}\xi\left(x_t + \int_t^{t_j} f_{q_{j-1}}(x_s, u_s) ds\right) \\ &\quad - \frac{\partial t_j}{\partial x_{t_j-}} f_{q_j}(x_\omega, u_\omega)\big|_{\omega=t_j} + \int_{t_j}^{\theta} \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x} d\omega \\ &= \frac{\partial \xi(z)}{\partial z}\bigg|_{z=x_t+\int_t^{t_j} f_{q_{j-1}}^{(x_s,u_s)} ds}^T \frac{\partial}{\partial x}\left(x_t + \int_t^{t_j} f_{q_{j-1}}^{(x_s,u_s)} ds\right) \\ &\quad - \frac{\partial t_j}{\partial x_{t_j-}} f_{q_j}(x_{t_j}, u_{t_j}) + \int_{t_j}^{\theta} \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x} d\omega\end{aligned} \tag{74}$$

which gives

$$\begin{aligned}\frac{\partial x_\theta}{\partial x} &= -\frac{\partial t_j}{\partial x_{t_j-}} f_{q_j}(x_{t_j}, u_{t_j}) + \int_{t_j}^{\theta} \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x} d\omega \\ &+ \nabla\xi|_{x_{t_j^-}}\left(I_{n\times n} + \frac{\partial t_j}{\partial x_{t_j-}} f_{q_{j-1}}^{(x_{t_j-}, u_{t_j-})}\right. \\ &\left. + \int_t^{t_j} \frac{\partial x_s}{\partial x}^T \frac{\partial f_{q_{j-1}}^{(x_s, u_s)}}{\partial x_s} ds\right).\end{aligned} \tag{75}$$

Note that in the above equations, the partial derivative $\frac{\partial t_j}{\partial x_{t_j-}}$ is not necessarily zero because for $\delta x_t \in \mathbb{R}^n$, the perturbed trajectory $x_s + \delta x_s$ arrives on the switching manifold m at a different time $t_j'- = (t_j + \delta t_j)-$, $\delta t_j \in \mathbb{R}$. This requires an arbitrary modification of the input, which we take to be

$$I_{L-j+1}' = ((t_j + \delta t, \sigma_j), u') \tag{76}$$

with

$$u_s' = \begin{cases} u_s, & s \in [t, t_j) \\ u(t_j-), & s \in [t_j, t_j + \delta t) \\ u_s, & s \in [t_j + \delta t, t_{j+1}) \end{cases} \tag{77}$$

if $\delta t \geq 0$ and

$$u'_s = \begin{cases} u_s, & s \in [t, t_j + \delta t) \\ u(t_j) \equiv u(t_j +), & s \in [t_j + \delta t, t_j) \\ u_s, & s \in [t_j, t_{j+1}) \end{cases} \tag{78}$$

if $\delta t < 0$. Since $I'_{L-j+1} = I_{L-j+1}$ holds everywhere except only on $[t_j, t_j + \delta t)$ (or $[t_j + \delta t, t_j)$ if $\delta t < 0$), the measure of the set of modified controls is of the order $|\delta t|$. Evidently, the perturbed trajectory arrives on the switching manifold when

$$m\left(x_{t_j+\delta t_j -} + \delta x_{t_j+\delta t_j -}\right) = 0. \tag{79}$$

For $\delta t \geq 0$, we may write

$$m\left(x_{t_j -} + \delta x_{t_j -} + \int_{t_j}^{t_j+\delta t} f_{q_{j-1}}\left(x_s + \delta x_s, u_{t_j -}\right)\right)$$
$$= m\left(x_{t_j -}\right) = 0 \tag{80}$$

which results in

$$\left[\nabla m\left(x_{t_j -}\right)\right]^T \left[\delta x_{t_j -} + f_{q_{j-1}}\left(x_{t_j -}, u_{t_j -}\right)\delta t + O\left(\delta t^2\right)\right] = 0 \tag{81}$$

or

$$\delta t = \frac{-\nabla m^T \delta x_{t_j -}}{\nabla m^T f_{q_{j-1}}\left(x_{t_j -}, u_{t_j -}\right)} + O\left(\delta t^2\right). \tag{82}$$

Equations (81) and (82) are derived similarly for $\delta t < 0$. In particular, as $t \uparrow t_j$ and $\theta \downarrow t_j$, (73) becomes

$$\frac{\partial J\left(t_j -, q_{j-1}, x_{t_j -}, L-j+1; I_{L-j+1}\right)}{\partial x_{t_j -}} = \frac{\partial t_j}{\partial x_{t_j -}} l_{q_{j-1}}^{\left(x_{t_j -}, u_{t_j -}\right)}$$
$$+ \int_{t_j -}^{t_j} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_{j-1}}(x_s, u_s)}{\partial x_s} ds + \left[\frac{\partial x_{t_j -}}{\partial x_{t_j -}}\right]^T \frac{\partial c\left(x_{t_j -}\right)}{\partial x_{t_j -}}$$
$$- \frac{\partial t_j}{\partial x_{t_j -}} l_{q_j}\left(x_{t_j}, u_{t_j}\right) + \int_{t_j}^{t_j +} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_j}\left(x_s, u_s\right)}{\partial x_s} ds$$
$$+ \left[\frac{\partial x_{t_j +}}{\partial x_{t_j -}}\right]^T \frac{\partial J\left(t_j +, q_j, x_{t_j +}, L-j; I_{L-j}\right)}{\partial x_{t_j +}} \tag{83}$$

or

$$\frac{\partial J\left(t_j -, q_{j-1}, x_{t_j -}, L-j+1; I_{L-j+1}\right)}{\partial x_{t_j -}}$$
$$= \frac{\partial t_j}{\partial x_{t_j -}} l_{q_{j-1}}\left(x_{t_j -}, u_{t_j -}\right) + \frac{\partial c\left(x_{t_j -}\right)}{\partial x_{t_j -}} - \frac{\partial t_j}{\partial x_{t_j -}} l_{q_j}\left(x_{t_j}, u_{t_j}\right)$$
$$+ \left[\frac{\partial x_{t_j +}}{\partial x_{t_j -}}\right]^T \frac{\partial J\left(t_j +, q_j, x_{t_j +}, L-j; I_{L-j}\right)}{\partial x_{t_j +}} \tag{84}$$

and (75) turns into

$$\frac{\partial x_{t_j +}}{\partial x_{t_j -}} = \nabla\xi|_{x_{t_j -}} - \frac{\partial t_j}{\partial x_{t_j -}}\left(f_{q_j}^{\left(x_{t_j}, u_{t_j}\right)} - \nabla\xi|_{x_{t_j -}} f_{q_{j-1}}^{\left(x_{t_j -}, u_{t_j -}\right)}\right). \tag{85}$$

Therefore

$$\frac{\partial J\left(t_j -, q_{j-1}, x_{t_j -}, L-j+1; I_{L-j+1}\right)}{\partial x_{t_j -}} =$$
$$\frac{-\partial t_j}{\partial x_{t_j -}}\left(\left(l_{q_j}\left(x_{t_j}, u_{t_j}\right) - l_{q_{j-1}}\left(x_{t_j -}, u_{t_j -}\right)\right)\right.$$
$$\left. + \left(f_{q_j} - \nabla\xi f_{q_{j-1}}\right)^T \frac{\partial J\left(t_j +, q_j, x_{t_j +}, L-j; I_{L-j}\right)}{\partial x_{t_j +}}\right)$$
$$+ \frac{\partial c\left(x_{t_j -}\right)}{\partial x_{t_j -}} + \nabla\xi^T \frac{\partial J\left(t_j +, q_j, x_{t_j +}, L-j; I_{L-j}\right)}{\partial x_{t_j +}}. \tag{86}$$

But in the limit as $\delta x_{t_j -} \in \mathbb{R}^n$ tends to zero, it is concluded from (82) that

$$\frac{\partial t_j}{\partial x_{t_j -}} = \frac{-\nabla m}{\nabla m^T f_{q_{j-1}}\left(x_{t_j -}, u_{t_j -}\right)}. \tag{87}$$

Therefore

$$\frac{\partial J\left(t_j -, q_{j-1}, x_{t_j -}, L-j+1; I_{L-j+1}\right)}{\partial x_{t_j -}}$$
$$= \nabla\xi^T \frac{\partial J\left(t_j +, q_j, x_{t_j +}, L-j; I_{L-j}\right)}{\partial x_{t_j +}} + \nabla c$$
$$+ \frac{l_{q_j} - l_{q_{j-1}} + f_{q_j} - \nabla\xi f_{q_{j-1}}{}^T \frac{\partial J t_j +, q_j, x_{t_j^+}, L-j; I_{L-j}}{\partial x_{t_j +}}}{\nabla m^T f_{q_{j-1}} x_{t_j -}, u_{t_j -}} \nabla m \tag{88}$$

This proves (37) with

$$p = \frac{\left(l_{q_j} - l_{q_{j-1}}\right) + \left(f_{q_j} - \nabla\xi f_{q_{j-1}}\right)^T \nabla_{x_{t_j +}} J^{\left(t_j +, q_j, x_{t_j +}, L-j; I_{L-j}^{[t_j, t_f]}\right)}}{\nabla m^T f_{q_{j-1}}\left(x_{t_j -}, u_{t_j -}\right)} \tag{89}$$

which is the same equation for p as in (38). Taking account of (71), and following a similar procedure as in part (i) of the proof, (35) is derived for $t \in (t_{j-1}, t_j]$, and as shown above, it is subject to the terminal and boundary conditions (36) and (37), respectively. This completes the proof. ■

Theorem 6.2: *Consider the hybrid system $\mathbb{H}$ together with Assumptions A0–A2 and the HOCP (11) for the hybrid cost (10). If there exists an optimal hybrid input with admissible set of discontinuities, then along each optimal trajectory, the adjoint process λ in the HMP and the gradient of the value function ∇V in the corresponding HDP satisfy the same family of differential equations, almost everywhere, i.e.,*

$$\frac{d}{dt}\nabla V = -\left[\frac{\partial}{\partial x} f_{q^o}\left(x^o, u^o\right)\right]^T \nabla V - \frac{\partial}{\partial x} l_{q^o}\left(x^o, u^o\right) \tag{90}$$

$$\frac{d}{dt}\lambda^o = -\left[\frac{\partial}{\partial x} f_{q^o}\left(x^o, u^o\right)\right]^T \lambda^o - \frac{\partial}{\partial x} l_{q^o}\left(x^o, u^o\right) \tag{91}$$

and satisfy the same terminal and boundary conditions, i.e.,

$$\nabla V\left(t_f, q^o, x^o_{q_L}\left(t_f\right), 0\right) = \nabla g\left(x^o_{q_L}\left(t_f\right)\right), \tag{92}$$

$$\begin{aligned}\nabla V &\left(t_j-, q^o_{j-1}, x^o_{q_{j-1}}\left(t_j-\right), L-j+1\right)\\ &= \nabla\xi|^T_{x^o_{q_{j-1}}(t_j-)}\nabla V\left(t_j+, q^o_j, x^o_{q_j}\left(t_j+\right), L-j\right)\\ &\quad + p\,\nabla m|_{x^o_{q_{j-1}}(t_j-)} + \nabla c|_{x^o_{q_{j-1}}(t_j-)}\end{aligned} \tag{93}$$

for the gradient of the value function, and

$$\lambda^o\left(t_f\right) = \nabla g\left(x^o_{q_L}\left(t_f\right)\right), \tag{94}$$

$$\begin{aligned}\lambda^o\left(t_j-\right) &= \nabla\xi|^T_{x^o_{q_{j-1}}(t_j-)}\lambda^o\left(t_j+\right) + p\,\nabla m|_{x^o_{q_{j-1}}(t_j-)}\\ &\quad + \nabla c|_{x^o_{q_{j-1}}(t_j-)}\end{aligned} \tag{95}$$

for the adjoint process. Hence, the adjoint process and the gradient of the value function are equal almost everywhere, i.e.,

$$\lambda^o = \nabla_x V \tag{96}$$

almost everywhere in the Lebesgue sense on $\bigcup_{i=0}^{L}[t_i, t_{i+1}]\times\mathbb{R}^{n_{q_i}}$. ■

Proof: Equations (91), (94), and (95) are direct results of the HMP in Theorem 4.1, and (90), (92), and (93) hold for the optimal feedback control having an admissible set of discontinuities because (35)–(37) hold for all feedback controls with admissible sets of discontinuities, including u^o corresponding to x^o. Hence, from Theorem 2.1 and the resulting uniqueness of the solutions of (90) and (91) that are identical almost everywhere on $t \in [t_0, t_f]$, it is concluded that (96) holds almost everywhere in the Lebesgue sense on $\bigcup_{i=0}^{L}[t_i, t_{i+1}]\times\mathbb{R}^{n_{q_i}}$. ■

VII. Examples

Example 1: Consider a hybrid system with the indexed vector fields:

$$\dot{x} = f_1\left(x, u\right) = x + x\,u \tag{97}$$

$$\dot{x} = f_2\left(x, u\right) = -x + x\,u \tag{98}$$

and the hybrid optimal control problem

$$\begin{aligned}J\left(t_0, t_f, h_0, 1; I_1\right) = &\int_{t_0}^{t_s}\frac{1}{2}u^2 dt + \frac{1}{1+\left[x\left(t_s-\right)\right]^2}\\ &+ \int_{t_s}^{t_f}\frac{1}{2}u^2 dt + \frac{1}{2}\left[x\left(t_f\right)\right]^2\end{aligned} \tag{99}$$

subject to the initial condition $h_0 = (q(t_0), x(t_0)) = (q_1, x_0)$ provided at the initial time $t_0 = 0$. At the controlled switching instant t_s, the boundary condition for the continuous state is provided by the jump map $x(t_s) = \xi(x(t_s-)) = -x(t_s-)$.

The HMP formulation and results: Writing down the HMP results for the above HOCP, the Hamiltonians are formed as

$$H_{q_1} = \frac{1}{2}u^2 + \lambda\,x\left(u+1\right) \tag{100}$$

$$H_{q_2} = \frac{1}{2}u^2 + \lambda\,x\left(u-1\right) \tag{101}$$

from which the minimizing control input for both Hamiltonian functions is determined as

$$u^o = -\lambda x\,. \tag{102}$$

Therefore, the adjoint process dynamics, determined from (19) and with the replacement of the optimal control input from (102), is written as

$$\dot{\lambda} = \frac{-\partial H_{q_1}}{\partial x} = -\lambda\left(u^o+1\right) = \lambda\left(\lambda\,x-1\right), \quad t\in\left(t_0, t_s\right) \tag{103}$$

$$\dot{\lambda} = \frac{-\partial H_{q_2}}{\partial x} = -\lambda\left(u^o-1\right) = \lambda\left(\lambda\,x+1\right), \quad t\in\left(t_s, t_f\right) \tag{104}$$

which are subject to the terminal and boundary conditions

$$\lambda\left(t_f\right) = \nabla g|_{x(t_f)} = x\left(t_f\right) \tag{105}$$

$$\begin{aligned}\lambda\left(t_s-\right) \equiv \lambda\left(t_s\right) &= \nabla\xi|_{x(t_s-)}\lambda\left(t_s+\right) + \nabla c|_{x(t_s-)}\\ &= -\lambda\left(t_s+\right) + \frac{-2x\left(t_s-\right)}{\left(1+\left[x\left(t_s-\right)\right]^2\right)^2}\,.\end{aligned} \tag{106}$$

The replacement of the optimal control input (102) in the continuous state dynamics (18) gives

$$\dot{x} = \frac{\partial H_{q_1}}{\partial\lambda} = x\left(1+u^o\right) = -x\left(\lambda\,x-1\right),\ t\in\left(t_0, t_s\right) \tag{107}$$

$$\dot{x} = \frac{\partial H_{q_2}}{\partial\lambda} = x\left(-1+u^o\right) = -x\left(\lambda\,x+1\right),\ t\in\left(t_s, t_f\right) \tag{108}$$

which are subject to the initial and boundary conditions

$$x\left(t_0\right) = x\left(0\right) = x_0 \tag{109}$$

$$x\left(t_s\right) = \xi\left(x\left(t_s-\right)\right) = -x\left(t_s-\right). \tag{110}$$

The Hamiltonian continuity condition (25) states that

$$\begin{aligned}H_{q_1}\left(t_s-\right) &= \frac{1}{2}\left[u^o\left(t_s-\right)\right]^2 + \lambda\left(t_s-\right)x\left(t_s-\right)\left[u^o\left(t_s-\right)+1\right]\\ &= \frac{1}{2}\left[-\lambda\left(t_s-\right)x\left(t_s-\right)\right]^2\\ &\quad + \lambda\left(t_s-\right)x\left(t_s-\right)\left[-\lambda\left(t_s-\right)x\left(t_s-\right)+1\right]\\ &= H_{q_2}\left(t_s+\right) = \frac{1}{2}\left[u^o\left(t_s+\right)\right]^2\\ &\quad + \lambda\left(t_s+\right)x\left(t_s+\right)\left[u^o\left(t_s+\right)-1\right]\\ &= \frac{1}{2}\left[-\lambda\left(t_s+\right)x\left(t_s+\right)\right]^2\\ &\quad + \lambda\left(t_s+\right)x\left(t_s+\right)\left[-\lambda\left(t_s+\right)x\left(t_s+\right)-1\right]\end{aligned} \tag{111}$$

which can be written, using (110), as

$$x\left(t_{s^-}\right)\left[\lambda\left(t_{s^-}\right) - \lambda\left(t_{s^+}\right)\right] = \frac{1}{2}\left[x\left(t_{s^-}\right)\right]^2\left[\left[\lambda\left(t_{s^-}\right)\right]^2 - \left[\lambda\left(t_{s^+}\right)\right]^2\right]. \tag{112}$$

The solution to the set of ODEs (103), (104), (107), (108) together with the initial condition (109) expressed at t_0, the terminal condition (105) determined at t_f, and the boundary conditions (110) and (106) provided at t_s, which is not *a priori* fixed but determined by the Hamiltonian continuity condition (112), determines the optimal control input and its corresponding optimal trajectory that minimize the cost $J(t_0, t_f, h_0, 1; I_1)$ over $\boldsymbol{I_1}$, the family of hybrid inputs with one switching. Interested readers are referred to [19], in which further steps are taken in order to reduce the above boundary value ODE problem into a set of algebraic equations using the special forms of the differential equations under study.

The HDP formulation and results: Theorem 5.5 states that the value function satisfies the HJB equation (31) almost everywhere. In particular

$$-\frac{\partial V(t, q_2, x, 0)}{\partial t} = \inf_u H_{q_2}\left(x, \frac{\partial V}{\partial x}, u\right)$$

$$= \inf_u \left\{ l_{q_2}(x,u) + \frac{\partial V}{\partial x} f_{q_2}(x,u) \right\} = \inf_u \left\{ \frac{1}{2}u^2 + \frac{\partial V}{\partial x}[-x + xu] \right\}$$

$$= \left\{ \frac{1}{2}u^2 + \frac{\partial V}{\partial x}[-x + xu] \right\}_{u = -x\frac{\partial V}{\partial x}}$$

$$= \frac{-1}{2}x^2\left(\frac{\partial V}{\partial x}\right)^2 - x\frac{\partial V}{\partial x} \tag{113}$$

and similarly

$$-\frac{\partial V(t, q_1, x, 1)}{\partial t} = \frac{-1}{2}x^2\left(\frac{\partial V}{\partial x}\right)^2 + x\frac{\partial V}{\partial x} \tag{114}$$

with the boundary conditions

$$V(t_f, q_2, x, 0) = g(x(t_f)) = \frac{1}{2}x^2 \tag{115}$$

for $V(t, q_2, x, 0)$, together with

$$V(t_s, q_1, x, 1) = \min_{\sigma \in \{\sigma_{q_1 q_2}\}} \left\{ V(t_s, q_2, -x, 0) + \frac{1}{1+x^2} \right\} \tag{116}$$

and

$$\frac{-1}{2}x^2\left(\frac{\partial V_{q_1}}{\partial x}\right)^2 + x\frac{\partial V_{q_1}}{\partial x} = \frac{-1}{2}(-x)^2\left(\frac{\partial V_{q_2}}{\partial x}\right)^2 - (-x)\frac{\partial V_{q_2}}{\partial x} \tag{117}$$

which determine $V(t, q_1, x, 1)$ and t_s.

The HMP—HDP relationship: In order to illustrate the results of Theorem 6.2, we first take the partial derivatives of (113) with respect to x to write

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t} - \frac{1}{2}x^2\left(\frac{\partial V}{\partial x}\right)^2 - x\frac{\partial V}{\partial x}\right) = 0 \tag{118}$$

or

$$\frac{\partial^2 V}{\partial x \partial t} - x\left(\frac{\partial V}{\partial x}\right)^2 - x^2\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} - x\frac{\partial^2 V}{\partial x^2} = 0. \tag{119}$$

It can easily be verified that the set of states with twice differentiability of $V(t, q_2, x, 0)$ is $M_{(2)} = (t_s, t_f) \times (\mathbb{R} - \{0\})$, which is open dense in $\mathbb{R} \times \mathbb{R}$, and therefore,

$$\frac{\partial^2 V}{\partial t \partial x} - x^2\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x^2} - x\frac{\partial^2 V}{\partial x^2} = x\left(\frac{\partial V}{\partial x}\right)^2 + \frac{\partial V}{\partial x}. \tag{120}$$

But from the definition of the total derivative, we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial V}{\partial x}\right) &= \frac{\partial^2 V}{\partial t \partial x} + \frac{\partial^2 V}{\partial x^2} f_{q_2}(x, u^o) \\ &= \frac{\partial^2 V}{\partial t \partial x} + \frac{\partial^2 V}{\partial x^2}\left(-x^2\frac{\partial V}{\partial x} - x\right) \\ &= \frac{\partial^2 V}{\partial t \partial x} - x^2\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x^2} - x\frac{\partial^2 V}{\partial x^2}. \end{aligned} \tag{121}$$

Therefore, from (120) and (121), the governing equation for $\nabla V(t, q_2, x, 0)$ is derived as

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}\right) = x\left(\frac{\partial V}{\partial x}\right)^2 + \frac{\partial V}{\partial x} = \frac{\partial V}{\partial x}\left(x\frac{\partial V}{\partial x} + 1\right) \tag{122}$$

which is the same as the dynamics (104) for $\lambda(t)$, $t \in (t_s, t_f)$.

Similarly, the differentiation of (114) results in

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial V}{\partial x}\left(x\frac{\partial V}{\partial x} - 1\right) \tag{123}$$

which is the same as the dynamics (103) for $\lambda(t)$, $t \in (t_0, t_s)$.

The equality of the terminal conditions for $\nabla V(t_f, q_2, x, 0)$ and $\lambda(t_f)$ becomes obvious by taking the gradient of (115), i.e.,

$$\frac{\partial V(t_f, q_2, x, 0)}{\partial x} = \frac{\partial g(x)}{\partial x} = x \tag{124}$$

which is equivalent to (105).

Moreover, the equality of the boundary conditions for $\nabla V(t_f, q_2, x, 0)$ and $\lambda(t_f)$ can be illustrated by taking the gradient of (116) and writing

$$\frac{\partial}{\partial x}V(t_s, q_1, x, 1) = \frac{\partial}{\partial x}\left(V(t_s, q_2, -x, 0) + \frac{1}{1+x^2}\right) \tag{125}$$

that gives

$$\frac{\partial V(t_s, q_1, x, 1)}{\partial x} = -\left.\frac{\partial V(t_s, q_2, y, 0)}{\partial y}\right|_{y=-x} + \frac{-2x}{(1+x^2)^2} \tag{126}$$

which is the same boundary condition as the boundary condition (106) for λ. Therefore, by the uniqueness of the results of the set of differential equations (122) and (123) for ∇V (or equivalently (104) and (103) for λ) with the terminal and boundary conditions (124) and (126) for ∇V (or equivalently (105) and (106) for λ), the gradient of the value function evaluated along every optimal trajectory is equal to the adjoint process corresponding to the same trajectory. Interested readers are referred to [19] for further discussion on this example. ■

Example 2: Consider the hybrid system with the indexed vector fields:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f_1(x, u) = \begin{bmatrix} x_2 \\ -x_1 + u \end{bmatrix} \tag{127}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f_2(x, u) = \begin{bmatrix} x_2 \\ u \end{bmatrix} \tag{128}$$

where autonomous switchings occur on the switching manifold described by

$$m(x_1(t_s-), x_2(t_s-)) \equiv x_2(t_s-) = 0 \tag{129}$$

with the continuity of the trajectories at the switching instant. Consider the hybrid optimal control problem defined as the minimization of the total cost functional

$$J = \int_{t_0}^{t_f} \frac{1}{2} u^2 dt + \frac{1}{2} \left(x_1 \left(t_s -\right)\right)^2 + \frac{1}{2} \left(x_2 \left(t_f\right) - v_{\mathrm{ref}}\right)^2 . \tag{130}$$

The HMP formulation and results: Employing the HMP, the corresponding Hamiltonians are defined as

$$H_1 = \lambda_1 x_2 + \lambda_2 \left(-x_1 + u\right) + \frac{1}{2} u^2 \tag{131}$$

$$H_2 = \lambda_1 x_2 + \lambda_2 u + \frac{1}{2} u^2 . \tag{132}$$

The Hamiltonian minimization with respect to u [see (24)] gives

$$u^o = -\lambda_2 \tag{133}$$

for both $q = 1$ and $q = 2$.

Therefore, the state dynamics (18) and the adjoint process dynamics (19) for $q = 1$ are derived as

$$\dot{x}_1 = \frac{\partial H_1}{\partial \lambda_1} = x_2 \tag{134}$$

$$\dot{x}_2 = \frac{\partial H_1}{\partial \lambda_2} = -x_1 + u^o = -x_1 - \lambda_2 \tag{135}$$

$$\dot{\lambda}_1 = \frac{-\partial H_1}{\partial x_1} = \lambda_2 \tag{136}$$

$$\dot{\lambda}_2 = \frac{-\partial H_1}{\partial x_2} = -\lambda_1 \tag{137}$$

and for $q = 2$ are derived as

$$\dot{x}_1 = \frac{\partial H_2}{\partial \lambda_1} = x_2 \tag{138}$$

$$\dot{x}_2 = \frac{\partial H_2}{\partial \lambda_2} = u^o = -\lambda_2 \tag{139}$$

$$\dot{\lambda}_1 = \frac{-\partial H_2}{\partial x_1} = 0 \tag{140}$$

$$\dot{\lambda}_2 = \frac{-\partial H_2}{\partial x_2} = -\lambda_1 \tag{141}$$

At the initial time $t = t_0$, the continuous valued states are specified by the initial conditions

$$x_1 \left(t_0\right) = x_{10} \tag{142}$$

$$x_2 \left(t_0\right) = x_{20} . \tag{143}$$

At the switching instant $t = t_s$, the boundary conditions for the states and adjoint processes are determined as

$$x_1 \left(t_s\right) = x_1 \left(t_s -\right) \equiv \lim_{t \uparrow t_s} x_1 \left(t\right) \tag{144}$$

$$x_2 \left(t_s\right) = x_2 \left(t_s -\right) = 0 \tag{145}$$

$$\lambda_1 \left(t_s\right) = \lambda_1 \left(t_s +\right) + \frac{\partial c}{\partial x_1} + p \frac{\partial m}{\partial x_1} = \lambda_1 \left(t_s +\right) + x_1 \left(t_s\right) \tag{146}$$

$$\lambda_2 \left(t_s\right) = \lambda_2 \left(t_s +\right) + \frac{\partial c}{\partial x_2} + p \frac{\partial m}{\partial x_2} = \lambda_2 \left(t_s +\right) + p . \tag{147}$$

And at the terminal time $t = t_f$, the adjoint processes are determined by (22) as

$$\lambda_1 \left(t_f\right) = \frac{\partial g}{\partial x_1} = 0 \tag{148}$$

$$\lambda_2 \left(t_f\right) = \frac{\partial g}{\partial x_2} = x_2 \left(t_f\right) - v_{\mathrm{ref}} . \tag{149}$$

Note that unlike t_0 and t_f, which are *a priori* determined, t_s is not fixed and needs to be determined by the Hamiltonian continuity condition (25) as

$$\begin{aligned} H_1 \left(t_s -\right) &= \lambda_1 \left(t_s -\right) x_2 \left(t_s -\right) - \lambda_2 \left(t_s -\right) x_1 \left(t_s -\right) \\ &- \frac{1}{2} \lambda_2 \left(t_s -\right)^2 = -\lambda_2 \left(t_s\right) x_1 \left(t_s -\right) - \frac{1}{2} \lambda_2 \left(t_s\right)^2 = H_2 \left(t_s +\right) \\ &= \lambda_1 \left(t_s +\right) x_2 \left(t_s +\right) - \frac{1}{2} \lambda_2 \left(t_s +\right)^2 = -\frac{1}{2} \lambda_2 \left(t_s +\right)^2 \end{aligned} \tag{150}$$

i.e.

$$\lambda_2 \left(t_s\right) x_1 \left(t_s -\right) + \frac{1}{2} \lambda_2 \left(t_s\right)^2 = \frac{1}{2} \lambda_2 \left(t_s +\right)^2 \tag{151}$$

that with the substitution of (147), it becomes

$$\left(\lambda_2 \left(t_s +\right) + p\right) x_1 \left(t_s -\right) + \frac{1}{2} \left(\lambda_2 \left(t_s +\right) + p\right)^2 = \frac{1}{2} \lambda_2 \left(t_s +\right)^2 . \tag{152}$$

The set of ODEs (134) to (141), together with the initial conditions (142) and (143) expressed at t_0, the boundary conditions (144)–(147) provided at t_s, and the terminal conditions (148) and (149) determined at t_f, with the two unknowns t_s and p determined by the Hamiltonian continuity condition (152) and the switching manifold condition (129), form an ODE boundary value problem whose solution results in the determination of the optimal control input and its corresponding optimal trajectory that minimize the cost $J(t_0, t_f, h_0, 1; I_1)$ over $\boldsymbol{I_1}$, the family of hybrid inputs with one switching on the switching manifold (129). Interested readers are referred to [21] for further steps taken in order to reduce the above boundary value ODE problem into a set of algebraic equations using the special forms of the differential equations under study.

The HDP formulation and results: For the linear differential equations (127) and (128), the Hamiltonians for the HJB equation are formed as

$$H_1 \left(x, \nabla V, u\right) = \frac{1}{2} u^2 + \frac{\partial V}{\partial x_1} \cdot x_2 + \frac{\partial V}{\partial x_2} \cdot \left(-x_1 + u\right) \tag{153}$$

$$H_2 \left(x, \nabla V, u\right) = \frac{1}{2} u^2 + \frac{\partial V}{\partial x_1} \cdot x_2 + \frac{\partial V}{\partial x_2} \cdot u \tag{154}$$

which have a minimizing control input

$$u^o = -\partial V / \partial x_2 . \tag{155}$$

Therefore, the HJB equations are expressed as

$$-\frac{\partial V \left(t, q_2, x, 0\right)}{\partial t} = \frac{-1}{2} \left(\frac{\partial V}{\partial x_2}\right)^2 + x_2 \frac{\partial V}{\partial x_1} \tag{156}$$

$$-\frac{\partial V \left(t, q_1, x, 1\right)}{\partial t} = \frac{-1}{2} \left(\frac{\partial V}{\partial x_2}\right)^2 + x_2 \frac{\partial V}{\partial x_1} - x_1 \frac{\partial V}{\partial x_2} . \tag{157}$$

The terminal condition at $t = t_f$ is specified as

$$V(t_f, q_2, x, 0) = \frac{1}{2}(x_2 - v_{\text{ref}})^2 \tag{158}$$

for $V(t, q_2, x, 0)$, and the boundary condition for $V(t, q_1, x, 1)$ and the switching instant $t = t_s$ are determined by

$$V(t_s, q_1, x, 1) = V(t_s, q_2, x, 0) + \frac{1}{2}x_1^2 \tag{159}$$

and

$$\frac{-1}{2}\left(\frac{\partial V_{q_1}}{\partial x_2}\right)^2 + x_2\frac{\partial V_{q_1}}{\partial x_1} - x_1\frac{\partial V_{q_1}}{\partial x_2} = \frac{-1}{2}\left(\frac{\partial V_{q_2}}{\partial x_2}\right)^2 + x_2\frac{\partial V_{q_2}}{\partial x_1} \tag{160}$$

subject to the switching manifold condition (129).

The HMP–HDP relationship: Similar to Example 1, in order to illustrate the result in Theorem 6.2, we shall take partial derivatives of (156) and (157) with respect to x. We note that by the definition of the total derivative

$$\frac{d}{dt}\left(\frac{\partial V(t, q_i, x, 2-i)}{\partial x}\right) = \frac{\partial^2 V}{\partial x \partial t} + \frac{\partial^2 V}{\partial x^2} f_{q_i}\left(x, -\frac{\partial V}{\partial x_2}\right) \tag{161}$$

which for $\nabla V(t, q_2, x, 0)$ it is equivalent to

$$\frac{d}{dt}\begin{bmatrix}\frac{\partial V(t,q_2,x,0)}{\partial x_1}\\ \frac{\partial V(t,q_2,x,0)}{\partial x_2}\end{bmatrix} = \begin{bmatrix}\frac{\partial^2 V}{\partial x_1 \partial t}\\ \frac{\partial^2 V}{\partial x_2 \partial t}\end{bmatrix} + \begin{bmatrix}\frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2}\\ \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2^2}\end{bmatrix}\begin{bmatrix}x_2\\ -\frac{\partial V}{\partial x_2}\end{bmatrix}$$
$$= \begin{bmatrix}\frac{\partial^2 V}{\partial x_1 \partial t} + x_2\frac{\partial^2 V}{\partial x_1^2} - \frac{\partial^2 V}{\partial x_1 \partial x_2}\frac{\partial V}{\partial x_2}\\ \frac{\partial^2 V}{\partial x_2 \partial t} + x_2\frac{\partial^2 V}{\partial x_2 \partial x_1} - \frac{\partial^2 V}{\partial x_2^2}\frac{\partial V}{\partial x_2}\end{bmatrix} \tag{162}$$

and for $\nabla V(t_s, q_1, x, 1)$ is

$$\frac{d}{dt}\begin{bmatrix}\frac{\partial V(t_s,q_1,x,1)}{\partial x_1}\\ \frac{\partial V(t_s,q_1,x,1)}{\partial x_2}\end{bmatrix}$$
$$= \begin{bmatrix}\frac{\partial^2 V}{\partial x_1 \partial t}\\ \frac{\partial^2 V}{\partial x_2 \partial t}\end{bmatrix} + \begin{bmatrix}\frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2}\\ \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2^2}\end{bmatrix}\begin{bmatrix}x_2\\ -x_1 - \frac{\partial V}{\partial x_2}\end{bmatrix}$$
$$= \begin{bmatrix}\frac{\partial^2 V}{\partial x_1 \partial t} + x_2\frac{\partial^2 V}{\partial x_1^2} - \frac{\partial^2 V}{\partial x_1 \partial x_2}\frac{\partial V}{\partial x_2} - x_1\frac{\partial^2 V}{\partial x_1 \partial x_2}\\ \frac{\partial^2 V}{\partial x_2 \partial t} + x_2\frac{\partial^2 V}{\partial x_2 \partial x_1} - \frac{\partial^2 V}{\partial x_2^2}\frac{\partial V}{\partial x_2} - x_1\frac{\partial^2 V}{\partial x_2^2}\end{bmatrix} \tag{163}$$

Taking the partial derivative of (156) with respect to x and making a substitution using the resulting equation in (162) yields

$$\frac{d}{dt}\left(\frac{\partial V(t, q_2, x, 0)}{\partial x_1}\right) = 0 \tag{164}$$

$$\frac{d}{dt}\left(\frac{\partial V(t, q_2, x, 0)}{\partial x_2}\right) = -\frac{\partial V(t, q_2, x, 0)}{\partial x_1} \tag{165}$$

which are equivalent to the differential equations (140) and (141) for $\lambda(t), t \in (t_s, t_f]$. Similarly, taking the partial derivative with respect to x of (157) and making a substitution in (163) gives

$$\frac{d}{dt}\left(\frac{\partial V(t, q_1, x, 1)}{\partial x_1}\right) = \frac{\partial V(t, q_1, x, 1)}{\partial x_2} \tag{166}$$

$$\frac{d}{dt}\left(\frac{\partial V(t, q_1, x, 1)}{\partial x_2}\right) = -\frac{\partial V(t, q_1, x, 1)}{\partial x_1} \tag{167}$$

which are equivalent to the differential equations (136) and (137) for $\lambda(t), t \in (t_0, t_s]$. Moreover, it can easily be verified that the optimal sensitivity process ∇V satisfies the terminal condition

$$\nabla V(t_f, q_2, x, 0) = \begin{bmatrix}\frac{\partial V(t_f,q_2,x,0)}{\partial x_1}\\ \frac{\partial V(t_f,q_2,x,0)}{\partial x_2}\end{bmatrix} = \begin{bmatrix}0\\ x_2(t_f) - v_{\text{ref}}\end{bmatrix} \tag{168}$$

and the boundary condition

$$\begin{bmatrix}\frac{\partial V(t_s,q_1,x,1)}{\partial x_1}\\ \frac{\partial V(t_s,q_1,x,1)}{\partial x_2}\end{bmatrix} = \begin{bmatrix}\frac{\partial V(t_s,q_2,x,0)}{\partial x_1}\\ \frac{\partial V(t_s,q_2,x,0)}{\partial x_2}\end{bmatrix} + \begin{bmatrix}\frac{\partial c(x(t_s-))}{\partial x_1}\\ \frac{\partial c(x(t_s-))}{\partial x_2}\end{bmatrix}$$
$$+ p\begin{bmatrix}\frac{\partial m(x(t_s-))}{\partial x_1}\\ \frac{\partial m(x(t_s-))}{\partial x_2}\end{bmatrix} = \begin{bmatrix}\frac{\partial V(t_s,q_2,x,0)}{\partial x_1} + x_1(t_s-)\\ \frac{\partial V(t_s,q_2,x,0)}{\partial x_2} + p\end{bmatrix} \tag{169}$$

subject to $x_2(t_s-) = 0$. Therefore, by the uniqueness of the results of the set of governing differential equations for ∇V and λ, which are subject to the same terminal and boundary conditions, along any optimal trajectory, the gradient of the value function is equal to the adjoint process corresponding to the same optimal trajectory. Interested readers are referred to [21] for further discussion on this example. ■

VIII. Concluding Remarks

In this paper, it is proved in the context of deterministic hybrid optimal control theory that the adjoint process in the HMP and the gradient of the value function in HDP are identical to each other almost everywhere along optimal trajectories; this is because they both satisfy the same Hamiltonian equations with the same boundary conditions. So due to the fact that a similar adjoint process–gradient process relationship holds for continuous parameter (i.e., nonhybrid) stochastic optimal control problems via the so-called Feynman–Kac formula (see, e.g., [6]), it is natural to expect the adjoint process in the stochastic HMP [61] and the gradient of the value function in stochastic HDP to be identical almost everywhere. Indeed, the formulation of stochastic HDP and the investigation of its relationship to the stochastic HMP is the subject of another study to be presented in a consecutive paper.

Appendix

A. Proof of Theorem 5.1

For simplicity of notation, we use $x, \hat{x}, u$, etc. instead of $x_q, \hat{x}_q, u_q$, etc. whenever the spaces $\mathbb{R}^{n_q}, \mathbb{R} \times \mathbb{R}^{n_q}, U_q$, etc., can easily be distinguished. For a given hybrid control input $I_{L-j+1} = (S_{L-j+1}, u)$, we use $\hat{x}_\tau \equiv \hat{x}(\tau; t, \hat{x}_t)$ to denote the extended continuous valued state as in (12) at the instant τ passing through x_t, where $t \le \tau \le t_f$. We also define

$$K_1 = \sup\left\{\left\|\hat{f}_q(\hat{x}, u)\right\| : (q, \hat{x}, u) \in Q \times \hat{B}_r \times U\right\} \tag{170}$$

where $\hat{B}_r := \{\hat{x} = [z, x^T]^T : |z|^2 + \|x\|^2 < r^2\}$.

Proof: First, consider the stage where no remaining switching is available and hence $t \in (t_L, t_{L+1}) = (t_L, t_f)$. In this case

$$\hat{x}(t_f; t, \hat{x}_t) = \hat{x}_t + \int_t^{t_f} \hat{f}_{q_L}(\hat{x}_\tau, u_\tau)\, d\tau \qquad (171)$$

which gives

$$\|\hat{x}(t_f; t, \hat{x}_t) - \hat{x}_t\| \leq K_1 |t_f - t| + \int_t^{t_f} \hat{K}_f \|\hat{x}(\tau; t, \hat{x}_t) - \hat{x}_t\|\, d\tau \qquad (172)$$

where $\hat{K}_f$ depends only on K_f and K_l which are defined in Assumptions A0 and A2, respectively. By the Gronwall–Bellman inequality, this results in

$$\begin{aligned}\|\hat{x}(t_f; t, \hat{x}_t) - \hat{x}_t\| &\leq K_1 |t_f - t| \\ &\quad + \int_t^{t_f} \hat{K}_f K_1 (\tau - t) e^{\hat{K}_f (t_f - \tau)} d\tau \\ &\leq K_2 |t_f - t| \leq K_2 |t_f - t_L| \end{aligned} \qquad (173)$$

where $K_2 = \max\{K_1, \hat{K}_f K_1 (t_f - t_L) e^{\hat{K}_f (t_f - t_L)}\}$. Hence, by the semigroup properties of ODE solutions and by use of (217), for $s \geq t$ and $\hat{x}_s \in N_{r_{\hat{x}}}(\hat{x}_t)$, we have

$$\begin{aligned}&\|\hat{x}(t_f; t, \hat{x}_t) - \hat{x}(t_f; s, \hat{x}_s)\| \leq \|\hat{x}_t - \hat{x}_s\| + \|\hat{x}(s; t, \hat{x}_t) - \hat{x}_t\| \\ &+ \int_s^{t_f} \hat{K}_f \|\hat{x}(\tau; t, \hat{x}_t) - \hat{x}(\tau; s, \hat{x}_s)\|\, d\tau \\ &\leq \|\hat{x}_t - \hat{x}_s\| + K_2 |s - t| + \int_s^{t_f} \hat{K}_f \|\hat{x}(\tau; t, \hat{x}_t) - \hat{x}(\tau; s, \hat{x}_s)\| d\tau \end{aligned} \qquad (174)$$

and therefore, by the Gronwall inequality, we have

$$\begin{aligned}&\|\hat{x}(t_f; t, \hat{x}_t) - \hat{x}(t_f; s, \hat{x}_s)\| \\ &\leq (\|\hat{x}_t - \hat{x}_s\| + K_2 |s - t|) e^{\hat{K}_f (t_f - s)} \\ &\leq (\|\hat{x}_t - \hat{x}_s\| + K_2 |s - t|) e^{\hat{K}_f (t_f - t_L)} \\ &\leq K \left(\|\hat{x}_t - \hat{x}_s\|^2 + |s - t|^2\right)^{\frac{1}{2}} \end{aligned} \qquad (175)$$

for some $K < \infty$ which depends only on $t_f - t_L$, K_1 and $\hat{K}_f$ and not on the control input.

Since $\hat{g}$ is Lipschitz in $\hat{x}$ and $\hat{x}(t_f; t, \hat{x}_t)$ is Lipschitz in $(t, \hat{x}_t) \equiv (t, [z_t, x_t^T]^T)$, the performance function

$$\begin{aligned}J(t, t_f, q, x, 0; I_0) &= \hat{g}(\hat{x}(t_f; t, \hat{x}_t)) \\ &\equiv \int_t^{t_f} l_q(x, u)\, ds + g(x_{q_L}(t_f)) \end{aligned} \qquad (176)$$

is Lipschitz in $x \in B_r$ (and $\hat{x} \in \hat{B}_r$) uniformly in $t \in (t_L, t_f)$ with a Lipschitz constant independent of the control. Furthermore, since the infimum of a family of Lipschitz functions with a common Lipschitz constant is also Lipschitz with the same Lipschitz constant, the value function $V(t, t_f, q, x, 0)$ with no switches remaining is Lipschitz in $x \in B_r$ uniformly in $t \in (t_L, t_f)$.

Now, consider $t, s \in (t_j, t_{j+1})$, where t_{j+1} indicates a time of an autonomous switching for the trajectory $\hat{x}(\tau; t, \hat{x}_t)$, and consider for definiteness the case where $\hat{x}(\tau; s, \hat{x}_s)$ arrives on the switching manifold described locally by $m(x) = 0$ at a later time $t_{j+1} + \delta t$ (the case with an earlier arrival time can be handled similarly by considering $\delta t < 0$). It directly follows by replacing $\hat{f}_{q_L}$ and t_f by $\hat{f}_{q_j}$ and $t_{j+1}-$ in the above arguments that

$$\begin{aligned}&\|\hat{x}(t_{j+1}-; t, \hat{x}_t) - \hat{x}(t_{j+1}-; s, \hat{x}_s)\| \\ &\qquad \leq K' \left(\|\hat{x}_t - \hat{x}_s\|^2 + |s - t|^2\right)^{\frac{1}{2}}. \end{aligned} \qquad (177)$$

Now, since

$$\begin{aligned}&\|\hat{x}(t_{j+1} + \delta t-; s, \hat{x}_s) - \hat{x}(t_{j+1}-; s, \hat{x}_s)\| \\ &\qquad \leq K_2 |t_{j+1} + \delta t - t_{j+1}| = K_2 |\delta t| \end{aligned} \qquad (178)$$

and

$$\begin{aligned}&\|\hat{x}(t_{j+1} + \delta t-; s, \hat{x}_s) - \hat{x}(t_{j+1}-; t, \hat{x}_t)\|^2 \\ &\leq \|\hat{x}(t_{j+1} + \delta t-; s, \hat{x}_s) - \hat{x}(t_{j+1}-; s, \hat{x}_s)\|^2 \\ &+ \|\hat{x}(t_{j+1}-; t, \hat{x}_t) - \hat{x}(t_{j+1}-; s, \hat{x}_s)\|^2 \end{aligned} \qquad (179)$$

it is sufficient to show that the upper bound for $|\delta t|$ is proportional to $(\|\hat{x}_t - \hat{x}_s\|^2 + |s - t|^2)^{\frac{1}{2}}$. This can be shown to hold by considering the fact that

$$\begin{aligned}&m(x(t_{j+1} + \delta t-; s, x_s)) \\ &= m\left(x(t_{j+1}-; s, x_s) + \int_{t_j}^{t_j + \delta t} f_{q_j}(x(\tau; s, x_s), u_{t_j -})\, d\tau\right) \\ &= m\left(x(t_{j+1}^-; t, x_t) + \delta x(t_{j+1}^-) + \int_{t_j}^{t_j + \delta t} f_{q_j}^{(x(\tau; s, x_s), u_{t_j^-})} d\tau\right) \\ &= m(x(t_{j+1}-; t, x_t)) = 0. \end{aligned} \qquad (180)$$

For $\|\delta x(t_{j+1}-)\| < \epsilon_{j+1}$ sufficiently small,

$$\nabla m^T \left(\delta x_{t_{j+1}-} + \int_{t_j}^{t_j + \delta t} f_{q_j}^{(x(\tau; s, x_s), u_{t_j -})} d\tau\right) + O(\epsilon_{j+1}^2) = 0 \qquad (181)$$

which is equivalent to

$$\begin{aligned}&\nabla m^T \delta x(t_{j+1}-) \\ &+ \int_{t_j}^{t_j + \delta t} \nabla m^T f_{q_j}(x(\tau; s, x_s), u_{t_j -}) d\tau + O(\epsilon_{j+1}^2) = 0. \end{aligned} \qquad (182)$$

Due to the transversal arrival of the trajectories with respect to the smooth switching manifold, $|\nabla m^T f_{q_j}|$ is lower bounded by a strictly positive number $k_{m,f}$ [see (2)], and hence,

$$\begin{aligned}&\left|\nabla m^T \delta x(t_{j+1}-) + O(\epsilon_{j+1}^2)\right| \\ &= \left|\int_{t_j}^{t_j + \delta t} \nabla m^T f_{q_j}^{(x(\tau; s, x_s), u_{t_j -})} d\tau\right| \\ &\geq \int_{t_j}^{t_j + \delta t} \left|\nabla m^T f_{q_j}^{(x(\tau; s, x_s), u_{t_j -})}\right| d\tau \geq k_{m,f} |\delta t| \end{aligned} \qquad (183)$$

which gives

$$|\delta t| \leq \frac{1}{k_{m,f}} \left(\|\nabla m\| \left\|\delta x\left(t_{j+1}-\right)\right\| + \left|O\left(\epsilon_{j+1}^2\right)\right|\right)$$
$$\leq \frac{1}{k_{m,f}}\|\nabla m\|\,\epsilon_{j+1} + \epsilon_{j+1} \leq \left(\frac{\|\nabla m\|}{k_{m,f}} + 1\right)\epsilon_{j+1} = K_{j+1}\epsilon_{j+1}. \tag{184}$$

Hence, for $t \in (t_j, t_{j+1})$ and $x_t \in B_r$, there exist a neighborhood $N_{r_x}(x_t)$ such that for $s \in (t_j, t_{j+1})$ and $x_s \in \mathcal{N}_{r_x}(x_t)$, we have $\|\delta x(t_{j+1}-)\| \leq K'(\|\hat{x}_t - \hat{x}_s\|^2 + |s-t|^2)^{\frac{1}{2}} < \epsilon_{j+1}$ in order to ensure that $\delta t \leq K_{j+1}\epsilon_{j+1}$, and consequently

$$\left\|\hat{x}\left(t_{j+1} + \delta t-; s, \hat{x}_s\right) - \hat{x}\left(t_{j+1}-; t, \hat{x}_t\right)\right\|$$
$$\leq K\left(\|\hat{x}_t - \hat{x}_s\|^2 + |s-t|^2\right)^{\frac{1}{2}} \tag{185}$$

for K independent of the control. Since $\hat{\xi}$ is smooth and time invariant, it is therefore Lipschitz in $\hat{x}$ uniformly in time. Also since $c(x(t_{J+1}))$ is embedded in $\hat{\xi}$ [see (15)], at the switching time t_{j+1}, we have

$$J\left(t_{j+1}-, q_j, \hat{x}, L-j; I_{L-j}\right)$$
$$= J\left(t_{j+1}, q_{j+1}, \hat{\xi}\left(\hat{x}\right),\ L-j-1; I_{L-j-1}\right) \tag{186}$$

the Lipschitz property for the cost to go function $J(t_{j+1}-, q_j, \hat{x}, L-j; I_{L-j})$ follows from the smoothness of $\hat{\xi}$ and the Lipschitz property of $J(t, q_{j+1}, \hat{x}_t, L-j-1; I_{L-j-1})$. Namely, by backward induction from the Lipschitzness of $J(t, q_L, \hat{x}_t, 0; I_o)$ proved earlier, it is concluded that $J(t, q_{L-1}, \hat{x}_t, 1; I_1)$ is Lipschitz, from which $J(t, q_{L-2}, \hat{x}_t, 1; I_2)$ is concluded to be Lipschitz, etc. Since the Lipschitz constant is independent of the control and because the infimum of a family of Lipschitz functions with a common Lipschitz constant is also Lipschitz with the same Lipschitz constant, (30) holds, and hence, the value function is Lipschitz. ■

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