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A Class of Linear Quadratic Gaussian Hybrid Optimal Control Problems with Realization–Independent Riccati Equations

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Abstract: A class of stochastic linear quadratic hybrid optimal control problems is presented for which the Hamiltonian boundary conditions appearing in the associated necessary optimality conditions of the Stochastic Hybrid Minimum Principle and Stochastic Hybrid Dynamic Programming are path-independent. Consequently, the linear quadratic Gaussian regulation problem associated with this class of stochastic hybrid optimal control problems can be solved via (stochastic) Riccati equations which are independent of the realization of stochastic diffusion terms. An analytic example of a scalar hybrid system is provided to illustrate the results, and the relation to the deterministic case is discussed.

Keywords: Hybrid Systems, Stochastic Control, Linear Quadratic Gaussian, Optimal Control

1. INTRODUCTION

Linear Quadratic (LQ) problems constitute an extremely important class of optimal control problems, since they can model many problems in applications, and more importantly, many nonlinear control problems can be reasonably approximated by the LQ problems. Moreover, solutions of LQ problems exhibit elegant properties due to their simple and nice structures. For deterministic linear quadratic optimal control problems, one can employ the elementary method of completion of squares and obtain an optimal control in a linear state feedback form via the so-called Riccati equation (see e.g. Yong and Zhou (1999)). Along this line, the solvability of the Riccati equation leads to that of the LQ problem. It is interesting to note that both the Minimum Principle (MP) by Pontryagin et al. (1962) and Dynamic Programming (DP) by Bellman (1966) can lead to the Riccati equation, by which one can see more clearly the relationship between MP and DP.

For stochastic LQ problems, which are also called Linear Quadratic Gaussian (LQG) problems, the method of completion of squares, the Stochastic Minimum Principle (SMP), and Stochastic Dynamic Programming (SDP) all give rise to a stochastic Riccati equation (see e.g. Yong and Zhou (1999)). This equation is quite different from the conventional Riccati equation arising in the deterministic LQ problems. One of the main differences between the stochastic differential equations appearing in stochastic optimal control problems and deterministic differential equations for deterministic problems is that "time" cannot be reversed and solvability is interpreted as the existence of solutions adapted solely to the forward filtration (see e.g. Ma and Yong (1999)). This requires the introduction of a notion of forward-backward stochastic differential equations (FBSDE), first presented by Bismut (1978), and then elaborated more in the optimal control framework by Bensoussan (1983), Pardoux and Peng (1990), etc., and in the general theory of forward-backward stochastic differential equations by Antonelli, Ma, Protter, Yong, Hu, and Peng (see e.g. Ma and Yong (1999) and references therein). Stochastic Dynamic Programming (SDP) including the Stochastic Hamilton-Jacobi-Bellman (SHJB) equation are presented by Kushner (1962), Krylov (2008), Fleming and Soner (2006), Fleming and Rishel (1975), Yong and Zhou (1999), and others. Versions of the Stochastic Minimum Principle (SMP) are presented by Kushner (1972); Kushner and Schweppe (1964), Haussmann (1986), Bismut (1978), Bensoussan (1982), and Peng (1990). It has been shown using both the SMP and SDP that a stochastic LQ problem is well-posed if there are solutions to the stochastic Riccati equation, and an optimal feedback control can then be obtained via these solutions. Although the stochastic LQ problem can be reduced to that of solving the stochastic Riccati equation, the existence and uniqueness of the solutions to the stochastic Riccati equation are generally available only for certain special cases (see e.g. Yong and Zhou (1999)).

The optimal control of stochastic hybrid systems, i.e. control systems that involve the interaction of continuous dynamics, discrete dynamics and stochastic diffusions, has been the subject of a limited number of studies. Versions of non-classical stochastic optimal control problems have been studied by Wu and Zhang (2011), Shi and Wu (2010a,b), etc. but the class of problems addressed lack many of the key features of hybrid systems, most notably changes in dynamics and costs. In the context of Stochastic Hybrid Dynamic Programming (SHDP), Bensoussan and Menaldi (2000) presents the optimality conditions for infinite horizon problems where optimal controls are stationary. In the context of the Stochastic Hybrid Hybrid Minimum Principle (SHMP) Aghayeva and Abushov (2011) presented the optimality conditions for controlled switchings only, and Pakniyat and Caines (2016b) presented the SHMP for a general class of stochastic hybrid systems where autonomous and controlled state jumps at switching instants are allowed to be accompanied by changes in the dimension of the state space. A feature of special interest in this work is the effect of

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hard constraints imposed by switching manifolds on diffusion-driven state trajectories.

In contrast to the stochastic case, the optimal control of deterministic hybrid systems has been the subject of numerous studies. The generalization of the fundamental Pontryagin Minimum Principle (PMP) results in the Hybrid Minimum Principle (HMP). The formulation by Clarke and Vinter (1989a,b), referred to by them as "Optimal Multiprocesses", provides a Minimum Principle for hybrid systems of a very general nature in which switching conditions are regarded as constraints in the form of set inclusions and the dynamics of the constituent processes are governed by (possibly nonsmooth) differential inclusions. A similar philosophy is followed by Sussmann (1999a,b) where a nonsmooth Minimum Principle is presented for hybrid systems possessing a general class of switching structures. Due to the generality of the results by Clarke and Vinter (1989a,b); Sussmann (1999a,b), degeneracy is not precluded and therefore, additional hypotheses need to be imposed to make the HMP results significantly informative (see e.g. Caines et al. (2006) for more discussion); such hypotheses (typically of a controllability nature) are usually too restrictive to cover many practical problems of engineering interest. An alternative philosophy, followed by Shaikh and Caines (2007), Garavello and Piccoli (2005), Taringoo and Caines (2011, 2013), and Pakniyat and Caines (2017b) is to ensure the validity of the HMP in a non-degenerate form by introducing hypotheses on the dynamics, transitions and switching events. To name a few other versions of the HMP in its appearances within the development of optimal control theory one cites the work of Lygeros et al. (1997), Riedinger et al. (1999), Xu and Antsaklis (2004), Azhmyakov et al. (2008), and Dmitruk and Kaganovich (2008).

The generalization of the method of Dynamic Programming for hybrid systems results in the theory of Hybrid Dynamic Programming (HDP). Infinite horizon - HDP formulations have been given by Bensoussan and Menaldi (1997), Branicky et al. (1998), Barles et al. (2010); Dharmatti and Ramaswamy (2005), as well as finite horizon HDP formulations appearing in the work of Hedlund and Rantzer (2002), Caines et al. (2007); Schöllig et al. (2007) and Shaikh and Caines (2009), to name but few of the major publications on the theory of HDP. The equivalence of the Hybrid Minimum Principle and Hybrid Dynamic Programming is established in Pakniyat and Caines (2014b, 2017b) by showing that under certain technical assumptions the adjoint process in the HMP and the gradient of the value function in HDP are governed by the same set of differential equations and have the same boundary conditions and hence are almost everywhere identical to each other along optimal trajectories.

In Pakniyat and Caines (2016b) the deterministic framework established in Pakniyat and Caines (2013, 2014a,b, 2017a,b) is extended in order to cover a general class of stochastic hybrid systems with state dependant diffusion fields which are subject to a large range of autonomous and controlled switchings and state jumps. First order variational analysis is performed on the stochastic hybrid optimal control problem via the needle variation methodology and the necessary optimality conditions are established in the form of the Stochastic Hybrid Minimum Principle (SHMP). In the absence of stochastic diffusions, it has been discussed in Pakniyat and Caines (2017b) that Riccati equations derived from the HMP and HDP for Linear Quadratic Hybrid Optimal Control Problems (LQ-HOCP) are, in general, path-dependent. This is due to the path-dependence of Hamiltonian boundary conditions appearing in the associated necessary conditions of the HMP and HDP. Subsequently, stochastic Riccati equations derived for Linear Quadratic Gaussian Hybrid Optimal Control Problems (LQG-HOCP) are generally realization-dependent as the necessary optimality conditions of the SHMP and Stochastic HDP (SHDP), including the Hamiltonian boundary conditions, generalize those in the deterministic case (see Pakniyat and Caines (2016b) and Pakniyat (2016) for more discussion).

In this paper, a class of Linear Quadratic Gaussian Hybrid Optimal Control Problems (LQG-HOCP) is presented for which the Hamiltonian boundary conditions are path-independent and therefore, the corresponding stochastic Riccati equations are independent from the realization of stochastic diffusion terms. An analytic example is provided to illustrate the results, and the relation to the deterministic case is discussed.

2. A CLASS OF LINEAR QUADRATIC GAUSSIAN HYBRID OPTIMAL CONTROL PROBLEMS

Let $(\Omega, \Im, P)$ be a probability space with filtration $\Im^t$, and $w(\mathbf{.})$ be a standard Wiener process. Consider a class of LQG-HOCP with completely observed states, i.e. $\Im^t = \sigma\{w(s) : 0 \le s \le t\}$, which is the natural filtration associated with the sigma-algebra generated by the Wiener process.

Consider a hybrid system possessing linear vector fields in the form of

$$dx_{q_i} = (A_{q_i} x_{q_i} + B_{q_i} u_{q_i})\, dt + G_{q_i} dw, \qquad t \in [t_i, t_{i+1}), \quad (1)$$

where $q_i \in Q$, $x_{q_i} \in \mathbb{R}^{n_{q_i}}$, $u_{q_i} \in \mathbb{R}^{m_{q_i}}$, $A_{q_i} \in \mathbb{R}^{n_{q_i} \times n_{q_i}}$, $B_{q_i} \in \mathbb{R}^{n_{q_i} \times m_{q_i}}$, $G_{q_i} \in \mathbb{R}^{n_{q_i}}$, $0 \le i \le L$, $t_{L+1} := tf$. The initial condition $(q,x)(t_0) = (q_0, x_0)$ is assumed to be deterministically known at the initial time t_0. In this paper, we only consider prearranged controlled switchings, which result in a fixed sequence of discrete states $q_1, q_2, \cdots, q_L$, and at the switching instances $t_j, 1 \le j \le L$, which are decision variables, i.e. not a priori determined, the state jump maps are provided as

$$x_{q_j}(t_j) = \Psi_{\sigma_j} x_{q_{j-1}}(t_j-) \equiv \Psi_{q_{j-1}q_j} x_{q_{j-1}}(t_j-). \quad (2)$$

Consider the LQG-HOCP for the hybrid cost

$$J = \frac{1}{2}\mathbb{E}\left\{ \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} \left\| x_{q_i}(t) \right\|^2_{L_{q_i}} + \left\| u_{q_i}(t) \right\|^2_{R_{q_i}} dt + \left\| x_{q_L}(t_f) \right\|^2_{H_{q_L}} \right\}, \quad (3)$$

where $0 \le L_{q_i}^T = L_{q_i} \in \mathbb{R}^{n_{q_i} \times n_{q_i}}$, $0 < R_{q_i}^T = R_{q_i} \in \mathbb{R}^{m_{q_i} \times m_{q_i}}$, $0 \le H_{q_L}^T = H_{q_L} \in \mathbb{R}^{n_{q_L} \times n_{q_L}}$.

It is further assume that

$$G_{q_k} = \Psi_{q_{k-1}q_k} G_{q_{k-1}}, \quad (4)$$

for all $1 \le k \le L$, which implies equivalent diffusion fields before and after switching events.

3. NECESSARY OPTIMALITY CONDITIONS OF THE STOCHASTIC HYBRID MINIMUM PRINCIPLE

In order to determine the necessary optimality conditions of the Stochastic Hybrid Minimum Principle (SHMP) established in Pakniyat and Caines (2016b), we form the family of system Hamiltonians as

$$H_q(x_q, u_q, \lambda_q, K_q) = \frac{1}{2}\left(\left\| x_q \right\|^2_{L_q} + \left\| u_q \right\|^2_{R_q} \right) + \lambda_q^T (A_q x_q + B_q u_q) + K_q^T G_q, \quad (5)$$

with $\lambda_q \in \mathbb{R}^{n_q}$, $K_q \in \mathbb{R}^{n_q}$. Then, according to the SHMP, for the optimal input u^o and the corresponding trajectory x^o there exists $\lambda^o, K_q^o : \mathfrak{I}^t$ – adapted, such that

$$H_q\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) \le H_q\left(x_q^o, v, \lambda_q^o, K_q^o\right), \tag{6}$$

almost everywhere $t \in [t_0, t_f]$, almost surely for all $v : \mathfrak{I}^t$ – measurable random variables in $\mathbb{R}^{m_q}$, that is to say the Hamiltonian is minimized with respect to the control input, which determines the optimal continuous (valued) input as

$$u_q^o = -R_q^{-1} B_q^T \lambda_q^o, \tag{7}$$

and further, the pairs of states and adjoint processes satisfy the following stochastic Hamiltonian canonical equations

$$\begin{aligned} dx_q{}^o &= \frac{\partial H_q}{\partial \lambda_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dt + \frac{\partial H_q}{\partial K_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dw \\ &= \left(A_q x_q^o - B_q R_q^{-1} B_q^T \lambda_q{}^o\right) dt + G_q dw, \end{aligned} \tag{8}$$

$$\begin{aligned} d\lambda_q{}^o &= -\frac{\partial H_{q^o}}{\partial x_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dt + K_q^o dw \\ &= -\left(L_{q_i} x_q^o + A_q^T \lambda_q^o\right) dt + K_q^o dw, \end{aligned} \tag{9}$$

almost everywhere $t \in [t_0, t_f]$, subject to

$$x_{q_0}^o(t_0) = x_0, \tag{10}$$

$$x_{q_j}^o(t_j) = \Psi_{\sigma_{q_{j-1}, q_j}} x_{q_{j-1}}^o(t_j -), \tag{11}$$

$$\lambda_{q_L}^o(t_f) = H_{q_L} x_{q_L}^o(t_f), \tag{12}$$

$$\lambda_{q_{j-1}}^o(t_j) = \left[\Psi_{\sigma_{q_{j-1}, q_j}}\right]^T \lambda_{q_j}^o(t_j +). \tag{13}$$

Moreover, at a switching time t_j the Hamiltonian satisfies

$$\begin{aligned} &H_{q_{j-1}}\left(x_{q_{j-1}}^o, u_{q_{j-1}}^o, \lambda_{q_{j-1}}^o, K_{q_{j-1}}^o\right) - K_{q_{j-1}}^{o^T} G_{q_{j-1}}\Big|_{t_j -} \\ &= H_{q_j}\left(x_{q_j}^o, u_{q_j}^o, \lambda_{q_j}^o, K_{q_j}^o\right) - K_{q_j}^{o^T} G_{q_j}\Big|_{t_j +}. \end{aligned} \tag{14}$$

□

4. HYBRID STOCHASTIC RICCATI EQUATIONS

We conjecture that $x_{q_i}^o$ and $\lambda_{q_i}^o$, $0 \le i \le L$, are related by

$$\lambda_{q_i}^o(t) = \Pi_{q_i}(t)\, x_{q_i}^o(t), \tag{15}$$

with $\Pi_{q_i}(t) \in C^1\left([t_i, t_{i+1}], \mathbb{R}^{n_{q_i} \times n_{q_i}}\right)$. Applying Itô's formula to (15) (see e.g. Yong and Zhou (1999)) with appropriate substitution of (8) to (13) (see also Pakniyat and Caines (2016a)) one obtains

$$K_{q_i}^o(t) = \Pi_{q_i}(t)\, G_{q_i}, \tag{16}$$

and

$$\dot{\Pi}_{q_i} = \Pi_{q_i} B_{q_i} R_{q_i}^{-1} B_{q_i}^T \Pi_{q_i} - \Pi_{q_i} A_{q_i} - A_{q_i}^T \Pi_{q_i} - L_{q_i}, \tag{17}$$

subject to

$$\Pi_{q_L}(t_f) = H_{q_L}, \tag{18}$$

$$\Pi_{q_{j-1}}(t_j) = \Psi_{\sigma_j}^T \Pi_{q_j}(t_j)\, \Psi_{\sigma_j}, \tag{19}$$

and from (14) we obtain

$$\begin{aligned} &x_{q_{j-1}}^{(t_j^-)^T}\left(L_{q_{j-1}} + \Pi_{q_{j-1}}^{(t_j)} A_{q_{j-1}} + A_{q_{j-1}}^T \Pi_{q_{j-1}}^{(t_j)} - \Pi_{q_{j-1}}^{(t_j)} B_{q_{j-1}} R_{q_{j-1}}^{-1} B_{q_{j-1}}^T \Pi_{q_{j-1}}^{(t_j)}\right) x_{q_{j-1}}^{(t_j^-)} \\ &= x_{q_j}^{(t_j)^T}\left(L_{q_j} + \Pi_{q_j}^{(t_j)} A_{q_j} + A_{q_j}^T \Pi_{q_j}^{(t_j)} - \Pi_{q_j}^{(t_j)} B_{q_j} R_{q_j}^{-1} B_{q_j}^T \Pi_{q_j}^{(t_j)}\right) x_{q_j}^{(t_j)}, \end{aligned} \tag{20}$$

for $1 \le j \le L$, where $\Pi_{q_j}(t_j) = \Pi_{q_j}(t_j +) \equiv \lim_{t \downarrow t_j} \Pi_{q_j}(t)$ since $\Pi_{q_i}(t) \in C^1\left([t_i, t_{i+1}], \mathbb{R}^{n_{q_i} \times n_{q_i}}\right)$.

Invoking (2) , (4) and (19), the Hamiltonian boundary condition (20) is reduced to the following *path-independent* expression

$$\begin{aligned} &L_{q_{j-1}} + \Psi_{\sigma_j}^T \Pi_{q_j}^{(t_j)} \Psi_{\sigma_j} A_{q_{j-1}} + A_{q_{j-1}}^T \Psi_{\sigma_j}^T \Pi_{q_j}^{(t_j)} \Psi_{\sigma_j} \\ &- \Psi_{\sigma_j}^T \Pi_{q_j}^{(t_j)} \Psi_{\sigma_j} B_{q_{j-1}} R_{q_{j-1}}^{-1} B_{q_{j-1}}^T \Psi_{\sigma_j}^T \Pi_{q_j}^{(t_j)} \Psi_{\sigma_j} \\ &= \Psi_{\sigma_j}^T \left(L_{q_j} + \Pi_{q_j}^{(t_j)} A_{q_j} + A_{q_j}^T \Pi_{q_j}^{(t_j)} - \Pi_{q_j}^{(t_j)} B_{q_j} R_{q_j}^{-1} B_{q_j}^T \Pi_{q_j}^{(t_j)}\right) \Psi_{\sigma_j}. \end{aligned} \tag{21}$$

Therefore, for the class of LQG-HOCP presented in Section 2, the set of Riccati equations (17), (18), (19) and (21) are realization independent.

5. ILLUSTRATIVE EXAMPLE

5.1 Problem Formulation

Consider the following scalar hybrid system for which the continuous state is governed by the following stochastic differential equations:

$$dx_1 = \left(\frac{31}{16} x_1 + u_1\right) dt + g_1 dw, \tag{22}$$

$$dx_2 = \left(\frac{3}{8} x_2 + u_2\right) dt + g_2\, dw, \tag{23}$$

with $g_1 = 1$, $g_2 = \sqrt{2} g_1 = \sqrt{2}$, and the performance measure is given as

$$\begin{aligned} J(t_0, t_f, h_0, L; I_L) := \mathbb{E}\Big\{ &\frac{1}{2}\int_{t_0}^{t_s}\left((u_1(t))^2 + \frac{1}{2}(x_1(t))^2\right) dt \\ &+ \frac{1}{2}\int_{t_s}^{t_f}\left((u_2(t))^2 + \frac{1}{4}(x_2(t))^2\right) dt + \frac{1}{2} \times 6\left(x_2(t_f)\right)^2 \Big\}, \end{aligned} \tag{24}$$

where t_s indicates the time of a controlled switching with the jump map $x_2(t_s) = \sqrt{2} x_1(t_s -)$.

5.2 Analytical Solution of the Riccati Equations

The associated stochastic Riccati equations in Section 4 are written as

$$\dot{\Pi}_1 = \Pi_1^2 - \frac{31}{8}\Pi_1 - \frac{1}{2} = (\Pi_1 - 4)\left(\Pi_1 + \frac{1}{8}\right), \tag{25}$$

$$\dot{\Pi}_2 = \Pi_2^2 - \frac{3}{4}\Pi_2 - \frac{1}{4} = (\Pi_2 - 1)\left(\Pi_2 + \frac{1}{4}\right), \tag{26}$$

which are subject to the terminal and boundary conditions

$$\Pi_2(t_f) = 6, \tag{27}$$

$$\Pi_1(t_s) = \left(\sqrt{2}\right)^2 \Pi_2(t_s) = 2\Pi_2(t_s), \tag{28}$$

$$(\Pi_1(t_s))^2 - \frac{31}{8}\Pi_1(t_s) - \frac{1}{2} = 2\left((\Pi_2(t_s))^2 - \frac{3}{4}\Pi_2(t_s) - \frac{1}{4}\right). \tag{29}$$

The above equations possess path-independent solutions in the form of

$$\Pi_2(t) = \frac{k_2 e^{\frac{-5}{4}t} + \frac{1}{4}}{k_2 e^{\frac{-5}{4}t} - 1}, \tag{30}$$

$$\Pi_1(t) = \frac{4 k_1 e^{\frac{-31}{8}t} + \frac{1}{8}}{k_1 e^{\frac{-31}{8}t} - 1}, \tag{31}$$

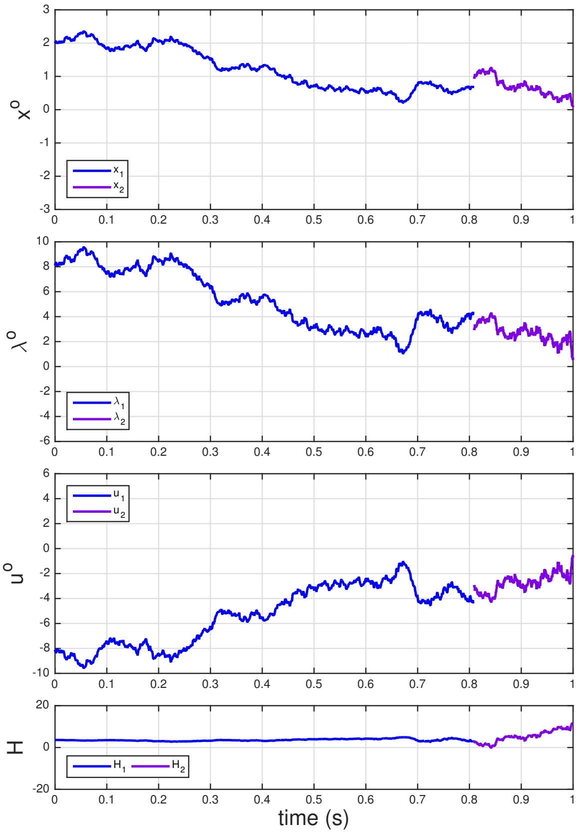

Fig. 1. A sample path for continuous states, adjoint processes, continuous inputs and Hamiltonians in the example with $t_f = 1$ and $x_0 = 2$, $g_1 = 1$, $g_2 = \sqrt{2}$.

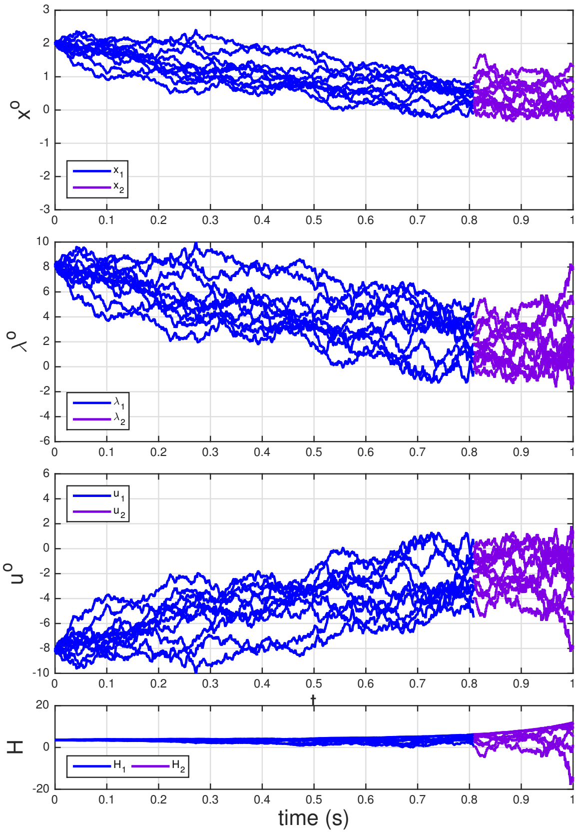

Fig. 2. Ten sample paths for continuous states, adjoint processes, continuous inputs and Hamiltonians in the example with $t_f = 1$ and $x_0 = 2$, $g_1 = 1$, $g_2 = \sqrt{2}$.

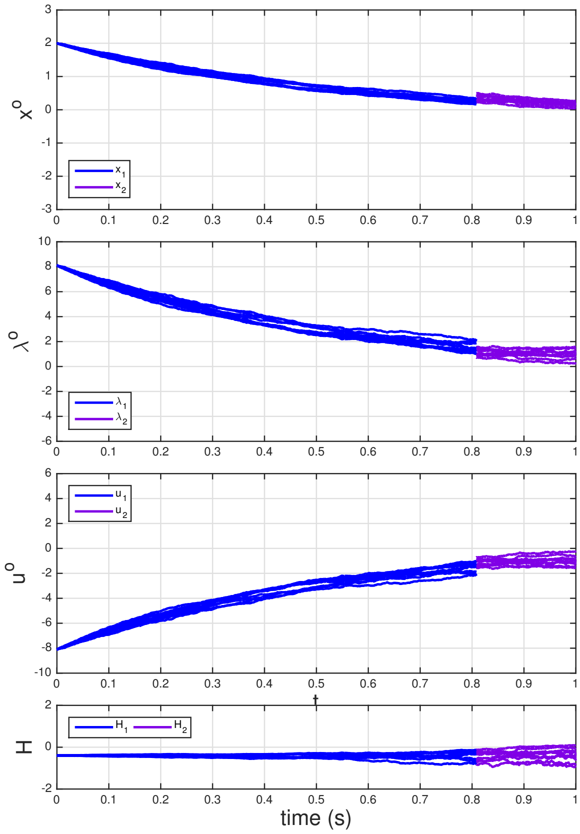

Fig. 3. Ten sample paths for continuous states, adjoint processes, continuous inputs and Hamiltonians in the example with $t_f = 1$ and $x_0 = 2$, $g_1 = 0.1$, $g_2 = 0.1\sqrt{2}$.

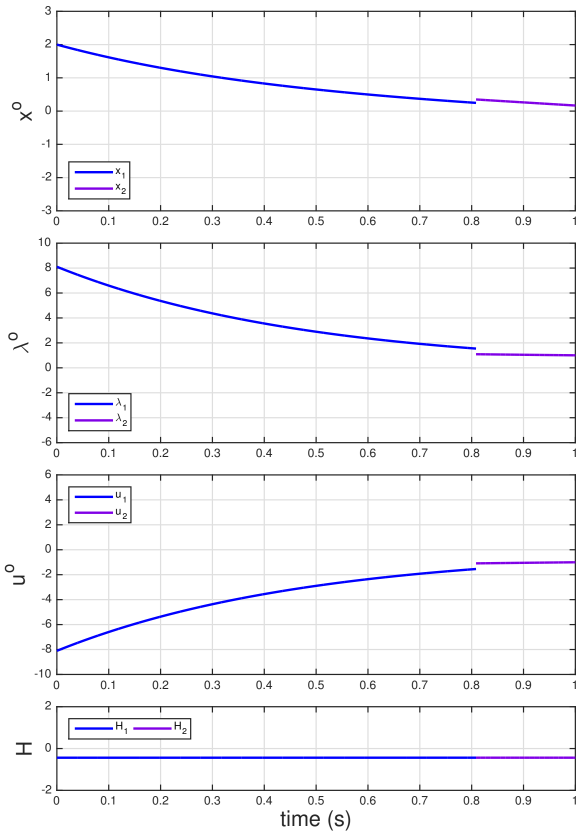

Fig. 4. The corresponding deterministic trajectories ($g_1 = g_2 = 0$) for the example with $t_f = 1$ and $x_0 = 2$.

where

$$k_2 = \frac{5}{4} e^{\frac{5}{4} t_f}, \tag{32}$$

$$t_s = \frac{4}{5} \ln\left(\frac{17}{27} k_2\right), \tag{33}$$

$$k_1 = \frac{17}{6} e^{\frac{33}{8} t_s}. \tag{34}$$

5.3 Simulations

For the values of $t_f = 1$, $x_0 = 2$, $g_1 = 1$, $g_2 = \sqrt{2}$, the simulations for a sample path of continuous states, adjoint processes, continuous inputs and Hamiltonians are illustrated in Figure 1. A collection of ten sample paths for continuous states, adjoint processes, continuous inputs and Hamiltonians for the same values is presented in Figure 2. For smaller values of diffusion coefficients g_1, g_2, trajectories are less influenced by diffusion terms and the results more closely resemble those of the deterministic LQ problem. This is illustrated in Figure 3 for the values of $g_1 = 0.1$, $g_2 = 0.1\sqrt{2}$ and in Figure 4 for the corresponding deterministic case with $g_1 = g_2 = 0$. It is observed in these figures that, in contrast to the deterministic case, Hamiltonian functions are not constants for stochastic hybrid optimal control problems.

6. CONCLUDING REMARKS

The linear quadratic Gaussian hybrid optimal control problems studied in this paper constitute a class of LQG-HOCP whose associated Riccati equations are independent form realizations of stochastic diffusions. In this paper we derive the (hybrid) stochastic Riccati equations using the Stochastic Hybrid Minimum Principle (SHMP). As proved in the case of deterministic hybrid optimal control problems (see e.g. Pakniyat and Caines (2014b, 2016a)), the adjoint process in the HMP and the gradient of the value function in Hybrid Dynamic Programming (HDP) are identical to each other almost everywhere. This intrinsic relation becomes an essential equivalence in the case of LQ-HOCPs. Due to the existence of the same relationship between the Stochastic Minimum Principle (SMP) and Stochastic Dynamic Programming (SDP) and the same equivalence in the LQG case (see e.g. Yong and Zhou (1999)), it is natural to expect the adjoint process in the SHMP and the gradient of the value function in Stochastic HDP (SHDP) to be identical almost everywhere. Indeed, the formulation of SHDP, the investigation of its relationship to the SHMP, and the demonstration of the equivalence of the SHMP and SHDP in the LQG-HOCP case is the subject of another study expected to be presented in a consecutive paper.

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