On the Minimum Principle and Dynamic Programming for Hybrid Systems

Ali Pakniyat [*] Peter E. Caines [**]

[*] Centre for Intelligent Machines (CIM) and the Department of
Electrical and Computer Engineering (ECE), McGill University
(e-mail: pakniyat@cim.mcgill.ca).
[**] Centre for Intelligent Machines (CIM) and the Department of
Electrical and Computer Engineering (ECE), McGill University
(e-mail: peterc@cim.mcgill.ca)

Abstract: Hybrid optimal control problems are studied for systems where autonomous and controlled state jumps are allowed at the switching instants and in addition to running costs, switching between discrete states incurs costs. A key aspect of the analysis is the relationship between the Hamiltonian and the adjoint process before and after the switching instants as well as the relationship between adjoint processes in the Minimum Principle and the value function in the Dynamic Programming. The results are illustrated through a simple, but yet very important, analytic example.

Keywords: Hybrid Systems, Optimal Control, Minimum Principle, Dynamic Programming

1. INTRODUCTION

There is now an extensive literature on the optimal control of hybrid systems (see e.g. [Barles et al. (2010); Bensoussan and Menaldi (1997); Branicky et al. (1998); Clarke and Vinter (1989); Dharmatti and Ramaswamy (2005); Lygeros (2004); Shaikh and Caines (2007); Sussmann (1999); Taringoo and Caines (2013, 2010); Xu and Antsaklis (2004)]). On one hand, the generalizations of the Pontryagin Maximum Principle (PMP), which is a necessary condition for optimality, results in the Hybrid Minimum Principle (HMP) [Garavello and Piccoli (2005a,b); Passenberg et al. (2011); Shaikh and Caines (2007); Sussmann (1999); Taringoo and Caines (2013, 2010); Xu and Antsaklis (2004)]. Given the initial conditions and a sequence of autonomous or controlled switchings, the HMP gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system. These conditions are expressed in terms of the minimization of the distinct Hamiltonians defined along the sequence of the discrete states of the hybrid trajectory. A feature of special interest is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Erdmann-Weierstrass conditions of the calculus of variations.

On the other hand, Dynamic Programming (DP) provides sufficient conditions for optimality based upon the Dynamic Programming Principle [Bellman (1966); Jacobson and Mayne (1970)]. With the exception of Hybrid Dynamic Programming (HDP) for regional dynamic systems [Caines et al. (2007); Schöllig et al. (2007)], the discretized version of HDP for continuous systems [Da Silva et al. (2012); Hedlund and Rantzer (2002)] and the verification theorem in [Shaikh and Caines (2009)], the current generalizations of Dynamic Programming to hybrid systems are formulated for systems that undergo jumps at autonomous and controlled switching times [Barles et al. (2010); Bensoussan and Menaldi (1997); Branicky et al. (1998); Dharmatti and Ramaswamy (2005)]. The assumed jump condition in HDP [Barles et al. (2010); Bensoussan and Menaldi (1997); Branicky et al. (1998); Dharmatti and Ramaswamy (2005)], which apparently is restrictive due to the obligation of the system to jump to a certain set, does not appear in the HMP formulation. In past work of the authors [Pakniyat and Caines (2013)] HMP results in the presence of switching costs were proved through the method of needle variations in the Mayer optimal control problem setup and was extended to the general Bolza setup through the calculus of variations methodology. In this paper, we first provide HMP results for the general case that includes autonomous and controlled switchings and jumps that may incur switching costs, and relate them to those of the HDP theorems; then we provide an analytic example illustrating the relationship between the HMP and HDP theories.

2. HYBRID SYSTEM

2.1 Basic Assumptions

A *hybrid system (structure)* $\mathbb{H}$ is a septuple

$$\mathbb{H} = \{H := Q \times \mathbb{R}^n, I := \Sigma \times U, \Gamma, A, F, \Xi, M\} \qquad (1)$$

where the symbols in the expression are defined as below.

A0: $Q = \{1, 2, ..., |Q|\} \equiv \{q_1, q_2, ..., q_{|Q|}\}, |Q| < \infty$, is a finite set of *discrete states (components)*.

$H := Q \times M$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$.

$I := \Sigma \times U$ is the set of system input values, where $|\Sigma| < \infty$.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete state transition map* which is the identity on the second ($\mathbb{R}^n$) component.

$\Xi : H \times \Sigma \to H$ is a time independent (partially defined) *continuous state jump transition map* which is the identity on the first (Q) component. All $\xi \in \Xi$ are assumed to be differentiable in the continuous state x.

$A : Q \times \Sigma \to Q$ denotes both a finite automaton and the automaton's associated transition function, on the state space Q and event set Σ, such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q-dependent subset $\{A(q, \sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : Q \times \mathbb{R}^n \times \Sigma \to H|_Q$. In other words, Γ can only make a discrete state transition in a hybrid state (q, x) if the automaton A can make the corresponding transition in q.

$U \subset \mathbb{R}^m$ is the set of *admissible input control values*, where U is an open bounded set in $\mathbb{R}^m$.

$\mathcal{U}(U) := L_\infty([t_0, T_*), U)$, which is the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*), T_* < \infty$. The boundedness property necessarily holds since admissible input functions take values in the open bounded set U which has compact closure $\bar{U}$.

F is an indexed collection of *vector fields* $\{f_q\}_{q \in Q}$ such that $f_q \in C^k(\mathbb{R}^n \times U \to \mathbb{R}^n), k \geq 1$ satisfies a uniform[x] Lipschitz condition, i.e. there exists $L_f < \infty$ such that $\|f_q(x_1, u) - f_q(x_2, u)\| \leq L_f \|x_1 - x_2\|, x_1, x_2 \in \mathbb{R}^n, u \in U, j \in Q$. We also assume that there exists $K_f < \infty$ such that $\max_{q \in Q} \left(\sup_{u \in U} (\|f_q(0, u)\|) \right) \leq K_f$.

$\mathcal{M} = \{\tilde{m}_\alpha^k : \alpha \in Q \times Q, k \in \mathbb{Z}_+\}$ denotes a collection of *switching manifold components*, also called *guard components*, such that, for any ordered pair $\alpha = (p, q)$, $\tilde{m}_\alpha^k$ is a smooth, i.e. C^∞ codimension 1 sub-manifold of $\mathbb{R}^n$, possibly with boundary $\partial\tilde{m}_\alpha^k$, described locally by $\tilde{m}_\alpha^k = \{x : \tilde{m}_\alpha^k(x) = 0\}$. It is assumed that $\tilde{m}_\alpha^k \cap \tilde{m}_\beta^k = \emptyset$, for all $\alpha, \beta \in Q \times Q, \alpha \neq \beta, k, l \in \mathbb{Z}_+$, except in those cases where, for some j, $\tilde{m}_\alpha^j$ is identified with its reverse ordered version $\tilde{m}_{\bar\alpha}^j$ giving $\tilde{m}_\alpha^j = \tilde{m}_{\bar\alpha}^j$.

This latter case corresponds to the common situation where the switch of vector fields on the passage of the continuous trajectory in one direction through a switching manifold is reversed if a reverse passage is performed by the continuous trajectory.

A *switching manifold* or *(autonomous discrete (state) transition) guard* $m_{p,q}$ (see Shaikh and Caines (2007)) is the union (over k) of a set of *switching manifold components* $m_{p,q}^k = \bigcup_{k_i} \tilde{m}_{p,q}^{k_i}; 1 \leq i \leq n(k)$ where,

(i) $\tilde{m}_{p,q}^{k_i}$ is a manifold component (as defined above)

(ii) $x \in \tilde{m}_{p,q}^k$ is such that $x \in \tilde{m}_{p,q}^{k_i} \cap \tilde{m}_{p,q}^{k_j}, k_i \neq k_j$ if and only if $x \in \partial\tilde{m}_{p,q}^{k_i} \cap \partial\tilde{m}_{p,q}^{k_j}$.

(iii) If $\partial\tilde{m}_{p,q}^{k_i} \cap \partial\tilde{m}_{p,q}^{k_j} \neq \emptyset$ then $\partial\tilde{m}_{p,q}^{k_i} \cap \partial\tilde{m}_{p,q}^{k_j}$ is a piecewise C^∞ codimension 2 sub-manifold of $\mathbb{R}^n$

A1: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0, q_j}(x_0) \neq 0$, for all $q_j \in Q$.

3. HYBRID OPTIMAL CONTROL PROBLEM

A2: Let $\{l_q\}_{q \in Q}, l_q \in C^{n_l}(\mathbb{R}^n \times U \to \mathbb{R}_+), n_l \geq 1$, be a family of cost functions; $\{c_\sigma\}_{\sigma \in \Sigma} \in C^{n_c}(\mathbb{R}^n \times \Sigma \to \mathbb{R}_+)$, $n_c \geq 1$, be a family of switching cost functions; and $g \in C^{n_g}(\mathbb{R}^n \to \mathbb{R}_+), n_g \geq 1$, be a terminal cost function satisfying the following:

There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $|l_q(x, u)| \leq K_l(1 + \|x\|^{\gamma_l}), x \in \mathbb{R}^n, u \in U, q \in Q$.

There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c_\sigma(x)| \leq K_c(1 + \|x\|^{\gamma_c}), x \in \mathbb{R}^n, \sigma \in \Sigma$.

There exists $K_g < \infty$ and $1 \leq \gamma_g < \infty$ such that $|g(x)| \leq K_g(1 + \|x\|^{\gamma_g}), x \in \mathbb{R}^n$.

Consider the initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and the upperbound of maximum number of swithchings $\bar{L} < \infty$. Let

$$S_L = \{(t_0, id), (t_1, \sigma_{q_0 q_1}), \ldots, (t_L, \sigma_{q_{L-1} q_L})\}$$
$$\equiv \{(t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L)\}$$

be a hybrid switching sequence and let $I_L := (S_L, u), u \in \mathcal{U}$, where $\mathcal{U} = \mathcal{U}^o$ or $\mathcal{U} = \mathcal{U}^{cpt}$, be a hybrid input trajectory which subject to A0 and A1 results in a (necessarily unique) hybrid state process (see Shaikh and Caines (2007)) and is such that $L + 2 < \bar{L}$ controlled and autonomous switchings occur on the time interval $[t_0, T(I_L)]$, where $T(I_L) \leq t_f$. In this paper, the number of switchings L is held fixed and we denote the corresponding set of inputs by $\{I_L\}$. Define the *hybrid cost function* on $[t_0, t_f]$ as

$$J(t_0, t_f, h_0, L; I_L) :=$$
$$\sum_{i=0}^{L} \int_{s=t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s)) ds \qquad (2)$$
$$+ \sum_{i=1}^{L} c_{\sigma_{q_{i-1} q_i}}(t_i, x_{q_i}(t_i-)) + g(x_{q_L}(t_f))$$

subject to

$$\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), a.e. \, t \in [t_i, t_{i+1}), \qquad (3)$$

$$h_0 = (q_0, x_{q_0}(t_0)) = (q_0, x_0), \qquad (4)$$

$$x_{q_j}(t_i) = \xi\left(x_{q_{i-1}}(t_i-)\right) \equiv \xi\left(\lim_{t \uparrow t_i} x_{q_{i-1}}(t)\right) \qquad (5)$$

where $1 \leq i \leq L$, $t_{L+1} = t_f < \infty$ and $L + 2 \leq \bar{L} < \infty$.

Then the Hybrid Optimal Control Problem (HOCP) is to find the infimum $J^o(t_0, t_f, h_0, L)$ over the family of input trajectories $\{I_L\}$, i.e.

$$J^o(t_0, t_f, h_0, L) = \inf_{I_L} J(t_0, t_f, h_0, L; I_L) \qquad (6)$$

4. HYBRID MINIMUM PRINCIPLE AND ITS RELATION TO HYBRID DYNAMIC PROGRAMMING

The Hybrid Minimum Principle (HMP) provides the necessary optimality conditions for the solution to the above optimal control problem. This results in a set of final value differential equations for the so-called adjoint processes which, together with the system's initial value differential equations and forwards discrete state transition functions, give the optimal trajectory and the optimal control input in case these are uniquely determined by the HMP necessary conditions.

In Hybrid Dynamic Programming, the value function V at a time $\tau \in [t_0, t_f]$ is naturally defined as the optimal cost-to-go for the hybrid system (1). Given the initial hybrid state h_0 at t_0 the Hybrid Dynamic Programming Theorem [Pakniyat and Caines (2014)] states that for any $q_j \in Q$ which corresponds to a time interval $(t_{j-1}, t_j]$, and for any $\tau \in (t_{j-1}, t_j]$, it is the case that

$$
\begin{aligned}
& V\left(t_0, q_0, x_0\right) \equiv J^o\left(t_0, t_f, h_0, L\right) \\
& = \inf_u \{ \sum_{i=0}^{j-2} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x, u\right) ds + \sum_{i=1}^{j-1} c_{\sigma_{q_{i-1} q_i}}\left(x_{q_i}\left(t_i-\right)\right) \\
& \quad + \int_{t_{j-1}}^{\tau} l_{q_{j-1}}\left(x, u\right) ds + V\left(\tau, q, x\left(\tau; u_{t_0}^\tau\right)\right) \}
\end{aligned}
\tag{7}
$$

where, setting $x(\tau) = x\left(\tau; u_{t_0}^\tau\right)$,

$$
\begin{aligned}
V\left(\tau, q, x\left(\tau\right)\right) = \inf_u \{ & \int_\tau^{t_j} l_{q_{j-1}}\left(x, u\right) ds \\
& + \sum_{i=j}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x, u\right) ds \\
& + \sum_{i=j}^{L} c_{\sigma_{q_{i-1} q_i}}\left(t_i, x_{q_i}\left(t_i-\right)\right) + g\left(x_{q_L}\left(t_f\right)\right) \}
\end{aligned}
\tag{8}
$$

Theorem 1 [Pakniyat and Caines (2014)] Consider the hybrid system $\mathbb{H}$ together with the assumptions A0, A1 and A2 as above and the HOCP (6). In addition, assume that in any discrete state $q_j \in Q$ the optimal trajectories for the system (3) are locally controllable on any time interval. Define the family of system Hamiltonians by

$$
H_q\left(x, \lambda, u\right) = \lambda^T f_q\left(x, u\right) + l_q\left(x, u\right)
\tag{9}
$$

$x, \lambda \in \mathbb{R}^n, u \in U, q \in Q$. Assume that the optimal control u^o is such that $u^o(t) \in U$ a.e. $t \in [t_0, t_f]$ and consider the optimal value for the cost function $J\left(t_0, t_f, h_0, L\right)$

$$
\begin{aligned}
J^o & \left(t_0, t_f, h_0, L\right) = \\
& \sum_{i=0}^{L} \int_{s=t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}^o\left(s\right), u^o\left(s\right)\right) ds \\
& + \sum_{i=1}^{L} c_{\sigma_{q_{i-1} q_i}}\left(t_i, x_{q_i}^o\left(t_i-\right)\right) + g\left(x_{q_L}^o\left(t_f\right)\right)
\end{aligned}
\tag{10}
$$

Then for the optimal input u^o and the corresponding optimal trajectory x^o, there exists an adjoint process λ^o for which

$$
\dot{\lambda}^o = -\frac{\partial H_{q^o}}{\partial x}\left(x^o, \lambda^o, u^o\right), \quad a.e. \ t \in [t_0, t_f],
\tag{11}
$$

$$
\lambda^o\left(t_f\right) = \nabla g\left(x^o\left(t_f\right)\right),
\tag{12}
$$

and

$$
\begin{aligned}
\lambda^o & \left(t_j-\right) \equiv \lambda^o\left(t_j\right) \\
& = \nabla \xi|_{x(t_j-)}^T \lambda^o\left(t_j+\right) + p \left.\nabla m\right|_{x(t_j-)} + \left.\nabla c\right|_{x(t_j-)},
\end{aligned}
\tag{13}
$$

with $p \in \mathbb{R}$ when t_j indicates the time of an autonomous switching, and $p = 0$ when t_j indicates the time of a controlled switching. Moreover, the Hamiltonian is minimized with respect to u which in the case of differentiability, it gives

$$
\left.\frac{\partial H_{q^o}}{\partial u}\left(x^o, \lambda^o, u\right)\right|_{u=u^o} = 0 \quad a.e. \ t \in [t_0, t_f],
\tag{14}
$$

and at a switching time t_j the Hamiltonian satisfies

$$
H_{q_{j-1}}\left(t_j-\right) = H_{q_j}\left(t_j+\right) + p\nabla_t m + \nabla_t c
\tag{15}
$$

Theorem 2 [Pakniyat and Caines (2014)] Under the assumptions of Theorem 1, with the additional assumptions that V, f_q and l_q are twice continuously differentiable, the adjoint process is almost everywhere equal to the gradient of the value function, i.e.

$$
\lambda^o = \nabla_x V \quad a.e. \ t \in [t_0, t_f]
\tag{16}
$$

together with the corresponding boundary conditions.

5. ILLUSTRATIVE EXAMPLE

In this section, we provide an example for the theorems above. Because of the complexity of nonlinear hybrid systems, not many analytic examples are available. The following example considers a scalar system that has an exponential growth in its first dynamics and an exponential decay in the second dynamics. The growth and the decay rates can be controlled by the control input at the expense of running costs. The (controlled) switching is decided freely by the controller (i.e. no switching manifold) while this switching incurs a state-dependent cost which is high near the origin, and hence the first dynamics is desirable before the switching. The terminal cost has a minimum at the origin and hence the second dynamics is desirable after the switching.

Consider a hybrid system with the following indexed vector fields

$$
\dot{x} = f_1\left(x, u\right) = x + x\, u,
\tag{17}
$$

$$
\dot{x} = f_2\left(x, u\right) = -x + x\, u
\tag{18}
$$

The initial condition $h_0 = \left(q\left(t_0\right), x\left(t_0\right)\right) = \left(1, x_0\right)$ and $t_0 = 0$ are assumed. The running costs $l_1\left(x, u\right)$ for $q = 1$ and $l_2\left(x, u\right)$ for $q = 2$ are equal to

$$
l_1\left(x, u\right) = l_2\left(x, u\right) = \frac{1}{2} u^2
\tag{19}
$$

At the moment of switching, the continuous state jumps according to

$$
x\left(t_s\right) = \xi\left(x\left(t_s-\right)\right) = -x\left(t_s-\right)
\tag{20}
$$

and the switching incurs cost given by

$$c\left(x\left(t_s-\right)\right) \equiv c_{1,2}\left(x\left(t_s-\right)\right) = \frac{1}{1+\left[x\left(t_s-\right)\right]^2} \quad (21)$$

The terminal cost function is defined as

$$g\left(x\left(t_f\right)\right) = \frac{1}{2}\left[x\left(t_f\right)\right]^2 \quad (22)$$

Hence, the HOCP for the sequence of switching states $(q_1, q_2) = (1, 2)$ will be

$$J\left(t_0, t_f, h_0, 1\right)$$
$$= \int_0^{t_s} \frac{1}{2}u^2 dt + \frac{1}{1+\left[x\left(t_s-\right)\right]^2} + \int_{t_s}^{t_f} \frac{1}{2}u^2 dt + \frac{1}{2}\left[x\left(t_f\right)\right]^2 \quad (23)$$

and the Hamiltonians are given by

$$H_1 = \frac{1}{2}u^2 + \lambda x\left(u+1\right) \quad (24)$$

$$H_2 = \frac{1}{2}u^2 + \lambda x\left(u-1\right) \quad (25)$$

Conditions on the Adjoint Processes and the Hamiltonians

Using (14) we get

$$u^o = -\lambda x \quad (26)$$

for both dynamics. The adjoint process dynamics is determined by (11) as

$$\dot{\lambda}_1 = -\lambda\left(u^o+1\right) = \lambda\left(\lambda x - 1\right) \quad (27)$$

$$\dot{\lambda}_2 = -\lambda\left(u^o-1\right) = \lambda\left(\lambda x + 1\right) \quad (28)$$

with the terminal values (from Eq. (12))

$$\lambda_2\left(t_f\right) = \nabla g|_{x\left(t_f\right)} = x\left(t_f\right) \quad (29)$$

and (from Eq. (13))

$$\lambda_1\left(t_s-\right) \equiv \lambda_1\left(t_s\right) = \nabla\xi|_{x\left(t_s-\right)}\lambda_2\left(t_s+\right) + \nabla c|_{x\left(t_s-\right)}$$
$$= -\lambda_2\left(t_s+\right) + \frac{-2x\left(t_s-\right)}{\left(1+\left[x\left(t_s-\right)\right]^2\right)^2} \quad (30)$$

As seen above, the adjoint process dynamics is coupled to the continuous state dynamics which, given the optimal control input, is coupled to the adjoint process:

$$\dot{x}_1 = x\left(1+u^o\right) = x\left(1-\lambda x\right) = -x\left(\lambda x - 1\right) \quad (31)$$
$$\dot{x}_2 = x\left(-1+u^o\right) = -x\left(1+\lambda x\right) = -x\left(\lambda x + 1\right) \quad (32)$$

with the initial values

$$x_1\left(0\right) = x_0 \quad (33)$$

and

$$x_2\left(t_s\right) = \xi\left(x_1\left(t_s-\right)\right) = -x_1\left(t_s-\right) \quad (34)$$

The Hamiltonian condition at the switching instant are given by Eq. (15) as

$$H_1\left(t_s-\right) = H_2\left(t_s+\right) \quad (35)$$

which gives

$$\frac{1}{2}\left[u^o\left(t_s-\right)\right]^2 + \lambda_1\left(t_s-\right)x\left(t_s-\right)\left[u^o\left(t_s-\right)+1\right]$$
$$= \frac{1}{2}\left[u^o\left(t_s+\right)\right]^2 + \lambda_2\left(t_s+\right)x\left(t_s+\right)\left[u^o\left(t_s+\right)-1\right] \quad (36)$$

$$\frac{1}{2}\left[-\lambda_1\left(t_s-\right)x\left(t_s-\right)\right]^2$$
$$+\lambda_1\left(t_s-\right)x\left(t_s-\right)\left[-\lambda_1\left(t_s-\right)x\left(t_s-\right)+1\right]$$
$$= \frac{1}{2}\left[-\lambda_2\left(t_s+\right)x\left(t_s+\right)\right]^2$$
$$+\lambda_2\left(t_s+\right)x\left(t_s+\right)\left[-\lambda_2\left(t_s+\right)x\left(t_s+\right)-1\right] \quad (37)$$

Putting $x\left(t_s+\right)$ from (34) in (37) and simplifying we get

$$x\left(t_s-\right)\left[\lambda_1\left(t_s-\right) - \lambda_2\left(t_s+\right)\right]$$
$$= \frac{1}{2}\left[x\left(t_s-\right)\right]^2\left[\left[\lambda_1\left(t_s-\right)\right]^2 - \left[\lambda_2\left(t_s+\right)\right]^2\right] \quad (38)$$

which requires that at least one of the following conditions hold

$$\begin{cases} x\left(t_s-\right) = 0 \\ \lambda_1\left(t_s-\right) = \lambda_2\left(t_s+\right) \\ x\left(t_s-\right)\left[\lambda_1\left(t_s-\right) + \lambda_2\left(t_s+\right)\right] = 2 \end{cases} \quad (39)$$

The first condition is impossible for $x_0 \neq 0$ as the control input cannot steer the trajectories to the origin. The third one is also a contradiction to Eq. (30) as the sum of the adjoint processes would need to be positive and negative at the same time. Hence

$$\lambda_1\left(t_s-\right) = \lambda_2\left(t_s+\right) \quad (40)$$

must hold which together with Eq. (30) gives

$$\lambda_1\left(t_s-\right) = \lambda_2\left(t_s+\right) = \frac{-x\left(t_s-\right)}{\left(1+\left[x\left(t_s-\right)\right]^2\right)^2} \quad (41)$$

Solving the set of differential equations (27), (28), (31) and (32) with the boundary conditions (29), (30), (33) and (34) constraint with (41) will give the optimal trajectories, the optimal adjoint processes as well as the optimal switching time ($\equiv$ switching state).

Derivation of the Adjoint Processes

Because of the special dynamics in this example, these equations can be solved analytically as follows:

$$\lambda_1' := \frac{d\lambda_1}{dx} = \frac{\dot{\lambda}_1}{\dot{x}_1} = \frac{\lambda\left(\lambda x - 1\right)}{-x\left(\lambda x - 1\right)} = \frac{-\lambda}{x} \quad (42)$$

which gives

$$\lambda_1 = \frac{\alpha}{x} \quad (43)$$

Similarly

$$\lambda_2 = \frac{\beta}{x} \quad (44)$$

Derivation of the Optimal Controls and Trajectories

Putting (43) and (44) in (26) we get

$$u^o\left(t\right) = -\alpha \quad t \in [0, t_s), \quad (45)$$
$$u^o\left(t\right) = -\beta \quad t \in [t_s, t_f] \quad (46)$$

Inserting (45) in (31) and (46) in (32) one gets

$$x\left(t\right) = x_0 e^{(1-\alpha)t} \quad t \in [0, t_s) \quad (47)$$

and

$$x\left(t\right) = x\left(t_s\right)e^{-(1+\beta)(t-t_s)}$$
$$= -x\left(t_s-\right)e^{-(1+\beta)(t-t_s)} \quad t \in [t_s, t_f] \quad (48)$$

where $x\left(t_s-\right)$ is replaced from (47) to give

$$x(t) = -x_0 e^{(1-\alpha)t_s - (1+\beta)(t-t_s)} \quad t \in [t_s, t_f] \qquad (49)$$

Therefore

$$\lambda_1(t) = \frac{\alpha}{x(t)} = \frac{\alpha}{x_0 e^{(1-\alpha)t}} \quad t \in [0, t_s] \qquad (50)$$

$$\lambda_2(t) = \frac{\beta}{x(t)} = \frac{-\beta}{x_0 e^{(1-\alpha)t_s - (1+\beta)(t-t_s)}} \quad t \in (t_s, t_f] \quad (51)$$

The condition (40) implies

$$\alpha = -\beta \qquad (52)$$

and (41) gives

$$\frac{\alpha}{x_0 e^{(1-\alpha)t_s}} = \frac{-x_0 e^{(1-\alpha)t_s}}{\left(1 + \left[x_0 e^{(1-\alpha)t_s}\right]^2\right)^2} \qquad (53)$$

or

$$\alpha = \frac{-x_0^2 e^{2(1-\alpha)t_s}}{\left(1 + x_0^2 e^{2(1-\alpha)t_s}\right)^2} \qquad (54)$$

and (29) results in

$$\frac{-\beta}{x_0 e^{(1-\alpha)t_s - (1+\beta)(t-t_s)}} \qquad (55)$$

$$= \frac{\alpha}{x_0 e^{(1-\alpha)(2t_s - t_f)}} = -x_0 e^{(1-\alpha)(2t_s - t_f)}$$

or

$$\alpha = -x_0^2 e^{2(1-\alpha)(2t_s - t_f)} \qquad (56)$$

Solving (54) and (56) will give α (and hence β) as well as t_s, considering that x_0 and t_f are given. The numerical results for $x_0 = 0.5$ and $t_f = 4$ are shown in the Figure 1.

The Value Function and its Gradient Processes

In order to illustrate the relation (16) we compute the value function. From the definition (8) we have:

$$V_2(t, x(t)) = g(x(t_f)) + \int_t^{t_f} \frac{1}{2}u^2 \quad t \in (t_s, t_f] \qquad (57)$$

and

$$V_1(t, x(t)) = g(x(t_f)) + c(x(t_s-)) + \int_t^{t_f} \frac{1}{2}u^2 \quad t \in [0, t_s] \qquad (58)$$

where (45) and (46) hold and (using $\beta = -\alpha$ where applicable):

$$x(t_f) = x(t) e^{-(1-\alpha)(t_f - t)} \quad t \in (t_s, t_f] \qquad (59)$$
$$x(t_f) = -x(t) e^{-(1-\alpha)(t + t_f - 2t_s)} \quad t \in [0, t_s] \qquad (60)$$
$$x(t_s-) = x(t) e^{(1-\alpha)(t_s - t)} \quad t \in [0, t_s) \qquad (61)$$

giving

$$V_2(t, x) = \frac{1}{2}x^2 e^{-2(1-\alpha)(t_f - t)} + \frac{1}{2}\alpha^2 (t_f - t) \quad t \in (t_s, t_f] \qquad (62)$$

and

$$V_1(t, x) = \frac{1}{2}x^2 e^{-2(1-\alpha)(t + t_f - 2t_s)}$$
$$+ \frac{1}{2}\alpha^2 (t_f - t) + \frac{1}{1 + x^2 e^{2(1-\alpha)(t_s - t)}} \quad t \in [0, t_s] \qquad (63)$$

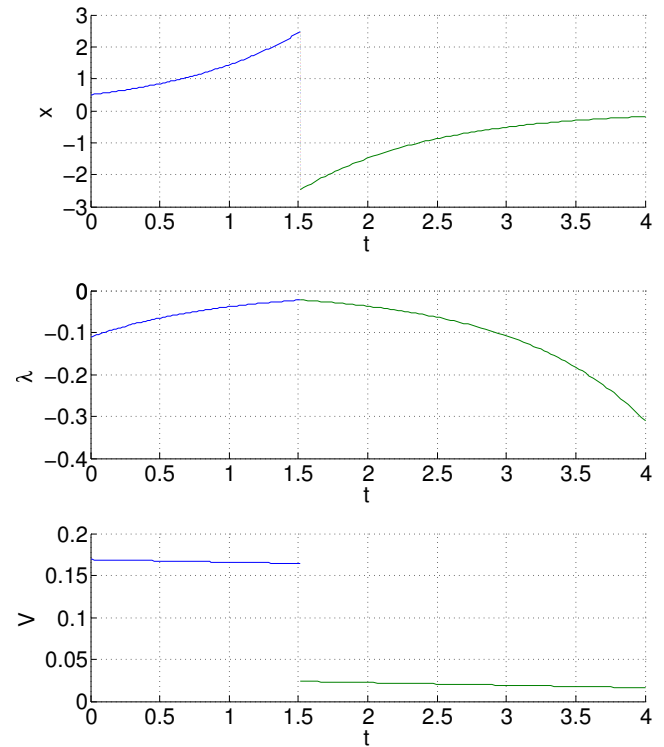Now, we first take the gradients of (62)

Fig. 1. The optimal trajectory (top), the adjoint process (middle) and the Value function (bottom) for the system in Example 1 with $x_0 = 0.5$ and $t_f = 4$

$$\nabla V_2(t, x) = x e^{-2(1-\alpha)(t_f - t)} = \frac{\left[x e^{-(1-\alpha)(t_f - t)}\right]^2}{x} \qquad (64)$$

But

$$\left[x e^{-(1-\alpha)(t_f - t)}\right]^2 = [x(t_f)]^2 \qquad (65)$$

and

$$[x(t_f)]^2 = x_0^2 e^{2(1-\alpha)(2t_s - t_f)} = -\alpha = \beta \qquad (66)$$

where the first equality in (66) is derived from (49) and the second one from (56). Inserting (66) into (64) gives

$$\nabla V_2(t, x) = \frac{\beta}{x} \equiv \lambda_2 \qquad (67)$$

The gradient of (63) also gives

$$\nabla V_1(t, x) = x e^{-2(1-\alpha)(t + t_f - 2t_s)} + \frac{-2x e^{2(1-\alpha)(t_s - t)}}{1 + x^2 e^{2(1-\alpha)(t_s - t)}} \qquad (68)$$

$$\nabla V_1(t, x) = \frac{1}{x}\left[x e^{-(1-\alpha)(t + t_f - 2t_s)}\right]^2$$
$$+ \frac{1}{x} \cdot \frac{-2\left[x e^{(1-\alpha)(t_s - t)}\right]^2}{\left(1 + \left[x e^{(1-\alpha)(t_s - t)}\right]^2\right)^2} \qquad (69)$$

Since

$$\left[x e^{-(1-\alpha)(t + t_f - 2t_s)}\right]^2 = [x(t_f)]^2 = -\alpha \qquad (70)$$

and

$$x e^{(1-\alpha)(t_s - t)} = x(t_s-) = x_0 e^{(1-\alpha)t_s} \qquad (71)$$

Eq. (69) gives

$$\nabla V_1\left(t, x\right) = \frac{-\alpha}{x} + \frac{1}{x} \cdot \frac{-2x_0^2 e^{2(1-\alpha)t_s}}{\left(1 + x_0^2 e^{2(1-\alpha)t_s}\right)^2} \qquad (72)$$

which according to Eq. (54) results in

$$\nabla V_1\left(t, x\right) = \frac{\alpha}{x} \equiv \lambda_1 \qquad (73)$$

Now, in order to illustrate (13), or equivalently (30), we evaluate (62) and (63) at $t = t_s$ to get

$$V_2\left(t_s+, x_{(t_s+)}\right) = \frac{1}{2}x_{(t_s+)}^2 e^{-2(1-\alpha)(t_f - t_s)} + \frac{1}{2}\alpha^2\left(t_f - t_s\right) \qquad (74)$$

and

$$V_1\left(t_s-, x_{(t_s-)}\right) = \frac{1}{2}x_{(t_s-)}^2 e^{-2(1-\alpha)(t_f - t_s)} \\ + \frac{1}{2}\alpha^2\left(t_f - t_s\right) + \frac{1}{1 + x_{(t_s-)}^2} \qquad (75)$$

Taking the gradient, one gets

$$\nabla V_2\left(t_s+, x_{(t_s+)}\right) = x_{(t_s+)} e^{-2(1-\alpha)(t_f - t_s)} = \lambda_2\left(t_s+\right) \qquad (76)$$

and

$$\nabla V_1\left(t_s-, x_{(t_s-)}\right) \\ = x_{(t_s-)} e^{-2(1-\alpha)(t_f - t_s)} + \frac{-2x_{(t_s-)}}{\left[1 + x_{(t_s-)}^2\right]^2} \\ = \lambda_1\left(t_s-\right) \qquad (77)$$

Noting that $x\left(t_s+\right) = -x\left(t_s-\right)$ the relation (30) is verified.

REFERENCES

Barles, G., Dharmatti, S., Ramaswamy, M., 2010. Unbounded viscosity solutions of hybrid control systems. ESAIM - Control, Optimisation and Calculus of Variations 16 (1), 176–193.

Bellman, R., 1966. Dynamic programming. Science 153 (3731), 34–37.

Bensoussan, A., Menaldi, J. L., 1997. Hybrid control and dynamic programming. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Application and Algorithm 3 (4), 395–442.

Branicky, M. S., Borkar, V. S., Mitter, S. K., 1998. A unified framework for hybrid control: Model and optimal control theory. IEEE Transactions on Automatic Control 43 (1), 31–45.

Caines, P., Egerstedt, M., Malhame, R., Schöllig, A., 2007. A hybrid bellman equation for bimodal systems. Proceedings of the 10th International Conference on Hybrid Systems: Computation and Control, HSCC 2007 4416 LNCS, 656–659.

Clarke, F. H., Vinter, R. B., 1989. Applications of optimal multiprocesses. SIAM Journal on Control and Optimization 27 (5), 1048–1071.

Da Silva, J. E., De Sousa, J. B., Pereira, F. L., 2012. Dynamic programming based feedback control for systems with switching costs. Proceedings of the IEEE International Conference on Control Applications, CCA 2012, 634–639.

Dharmatti, S., Ramaswamy, M., 2005. Hybrid control systems and viscosity solutions. SIAM Journal on Control and Optimization 44 (4), 1259–1288.

Garavello, M., Piccoli, B., 2005a. Hybrid necessary principle. SIAM Journal on Control and Optimization 43 (5), 1867–1887.

Garavello, M., Piccoli, B., 2005b. Hybrid necessary principle. Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, CDC-ECC '05 2005, 723–728.

Hedlund, S., Rantzer, A., 2002. Convex dynamic programming for hybrid systems. IEEE Transactions on Automatic Control 47 (9), 1536–1540.

Jacobson, D., Mayne, D., 1970. Differential dynamic programming. Modern Analytic and Computational Methods in Science and Mathematics, Volume 24.

Lygeros, J., 2004. On reachability and minimum cost optimal control. Automatica 40 (6), 917–927.

Pakniyat, A., Caines, P. E., 2013. The hybrid minimum principle in the presence of switching costs. Proceedings of the 52nd IEEE Conference on Decision and Control, CDC 2013, 3831 – 3836.

Pakniyat, A., Caines, P. E., 2014. On the minimum principle and dynamic programming for hybrid systems. Research Report (in preparation), Department of Electrical and Computer Engineering (ECE), McGill University, March 2014.

Passenberg, B., Leibold, M., Stursberg, O., Buss, M., 2011. The minimum principle for time-varying hybrid systems with state switching and jumps. Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, CDC-ECC 2011, 6723–6729.

Schöllig, A., Caines, P. E., Egerstedt, M., Malhamé, R., 2007. A hybrid bellman equation for systems with regional dynamics. Proceedings of the 46th IEEE Conference on Decision and Control 2007, CDC, 3393–3398.

Shaikh, M. S., Caines, P. E., 2007. On the hybrid optimal control problem: Theory and algorithms. IEEE Transactions on Automatic Control 52 (9), 1587–1603.

Shaikh, M. S., Caines, P. E., 2009. A verification theorem for hybrid optimal control problem. Proceedings of the IEEE 13th International Multitopic Conference, INMIC 2009.

Sussmann, H. J., 1999. Maximum principle for hybrid optimal control problems. Proceedings of the 38th IEEE Conference on Decision and Control (CDC), 425–430.

Taringoo, F., Caines, P., 2013. On the optimal control of impulsive hybrid systems on riemannian manifolds. SIAM Journal on Control and Optimization 51 (4), 3127–3153.

Taringoo, F., Caines, P. E., 2010. Gradient-geodesic hmp algorithms for the optimization of hybrid systems based on the geometry of switching manifolds. Proceedings of the 49th IEEE Conference on Decision and Control, CDC 2010, 1534–1539.

Xu, X., Antsaklis, P. J., 2004. Optimal control of switched systems based on parameterization of the switching instants. IEEE Transactions on Automatic Control 49 (1), 2–16.