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A MODIFICATION ON PERFORMANCE OF MEMS GYROSCOPES BY PARAMETRO-HARMONIC EXCITATION

Ali Pakniyat **Hassan Salarieh**

Gholamreza Vossoughi **Aria Alasty**

Center of Excellence in Design, Robotics and Automation (CEDRA),
School of Mechanical Engineering, Sharif University of Technology, Tehran, Iran

ABSTRACT

In this paper, parametric excitation for MEMS gyroscope proposed by Oropeza-Ramos, et al. [1-4] is examined and problems associated with this kind of excitation are shown. It is proved that origin has exponential stability for some sets of parameter values (including those considered in [1-4]). This stability is shown to be global for linearized system and local for the general nonlinear system. Hence, it is concluded that if there would be a periodic orbit, the system has difficulties reaching it. As a solution, a harmonic term to the parametric excitation is added and the new actuation is referred to as *parametro-harmonic excitation*. It is shown that there are some parameter values for which a stable periodic orbit exists.

Finally, stability of periodic orbit of the linear parametro-harmonically excited MEMS gyroscope is analyzed based on Floquet Theory. Figures show that in the non-resonant driving frequencies, only stiffness and damping play important roles in the stability of periodic response and other terms like excitation voltage and imposed external rotation are of less influence on this stability. However, in the parametric resonant regions, not only stiffness and damping affect stability, but also excitation voltage is of great importance.

INTRODUCTION

Gyroscopes can be classified in three categories: *Rotating*, *Optic* and *Vibrating* Gyroscopes. Rotating gyros, which are the oldest type, are massive and need precisely accurate manufacturing process. Optic gyros including *Ring-Laser* and *Fiber-Optic* Gyroscopes are too large and too expensive to be used broadly in many applications. However, the Vibrating-Type MEMS Gyroscopes are smaller, lighter, and cheaper. They need less power, have higher reliability, and can be multi-functional. These devices are so small that can be integrated with the electronic circuits needed for their operation.

Various types has been proposed and used for MEMS Gyroscopes. For instance *Tuning Forks*, *Ring-Type* and *Cantilever-Type* gyros are some examples of these types. In all vibrating gyros, there is an actuation which causes a part of the sensor vibrate in one direction called *drive mode* and one or more directions called *sense mode(s)*. Sense mode is ideally decoupled from the drive mode and only when there is an external rotation, becomes coupled to the drive mode by the resulting Coriolis force. In almost all of the currently fabricated gyros, the actuation mechanism is a simple harmonic voltage. One problem associated with harmonic excitation is that to have a significantly high amplitude at the sense mode(s), the resonant frequencies in all modes should be matched [5]. This requires an accurate control on sensor's dimensions which is hard to implement due to current fabrication processes and also may contribute to temperature drift problems if the natural frequencies don't track similarly with temperature [1, 5].

In order to overcome this problem, a number of methods have been proposed. Some of them rely on an improvement over design of the gyro. For instance, Alper, et al. [6-7] suggested some manufacturing tips such as adding an extra set of combs to electrostatically tune the frequencies, and Liu, et al. [8] improved spring beams by optimizing the shape of the suspension beams by a Cellular Automata approach. Other methods are based on using some closed loop control techniques to match the modes. For example, one can refer to Sung, et al. [9-10] and Chang et al. [11].

While improvements on design methods impose some financial burden on the fabrication process and closed loop control methods require additional electronics and a higher expense, a new method based on changing the law of excitation voltage has been proposed by Oropeza-Ramos, et al. [1-4]. By using parametric resonance as an actuation mechanism, not only a precise manufacturing process is not required, but also there is no need for the extra electronic circuits.

However, as shown in this paper, the parametrically excited system faces some fundamental problems: First, the nonlinear terms cannot be neglected in design and analysis because otherwise the origin becomes either globally exponentially stable or globally unstable. Furthermore, even when the nonlinear terms are considered, the same happens to the origin locally. Hence, for parameter values that the periodic response is stable, leaving the zero state (as the initial condition) to reach the periodic orbit confronts some challenging problems. We show that by adding a harmonic term to the parametric excitation, we can overcome these problems.

The following section presents the model that Oropeza-Ramos, et al considered to implement parametric resonance.

STATEMENT OF THE MODEL

In this paper a gyroscope with a single lumped mass is considered. The simplified equivalent model of the structure of this device is shown in Fig. 1 which is the same as the model used by Oropeza-Ramos, et al.

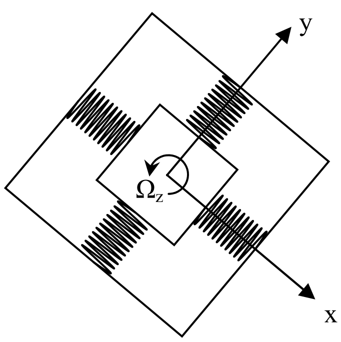

Fig. 1: A SIMPLIFIED MODEL OF THE CONSIDERED GYROSCOPE

The equation of motion for the system shown is presented in Eq. 1 where m is the mass, c_x and c_y are the damping coefficients, $F_r(x)$ and $F_r(y)$ represent the elastic restoring forces provided by the springs in the x and y direction respectively and $F_a(x,t)$ corresponds to the actuation force.

$$\begin{cases} m\ddot{x} + c_x\dot{x} + F_r\left(x\right) = F_a\left(x,t\right) + 2m\Omega_z\dot{y} \\ m\ddot{y} + c_y\dot{y} + F_r\left(y\right) = -2m\Omega_z\dot{x} \end{cases} \tag{1}$$

In this equation the terms $2m\Omega_z\dot{x}$ and $2m\Omega_z\dot{y}$ represent the rotation-induced Coriolis forces and the angular velocity is considered to be constant. The restoring force from the mechanical springs behaves as in Eq. 2.

$$F_r\left(\left[.\right]\right) = k_1\left[.\right] + k_3\left[.\right]^3 \tag{2}$$

In order to generate a parametric resonance response, the actuation force is produced by a set of non-interdigitated comb-fingers described by Oropeza-Ramos, et al. [1-4] as

$$F_a\left(x,t\right) = -r\left(x\right)V\left(t\right)^2 \tag{3}$$

where in their work, they considered the following form for the function $r(x)$.

$$r\left(x\right) = r_1 x + r_3 x^3 \tag{4}$$

where r_1 and r_3 are electrostatic coefficients that depend on the physical dimensions and spacing of the electrostatic comb-drives, and $V(t)$ is the voltage applied across the drives [2-5]. Since the electrostatic force has a square dependence on the applied voltage it is proposed to use a square rooted sinusoidal signal $V\left(t\right) = V_A\left(1+\cos\left(\omega t\right)\right)^{1/2}$ in order to isolate the parametric effects from the harmonic response [1-4]. Substituting F_a and F_r in Eq. 1 and re-scaling parameters, the gyroscopic system is described by

$$\begin{cases} x'' + \alpha_x x' + \left(\delta_x + 2\beta_x\cos 2\tau\right)x \\ \quad + \left(\delta_{x3} + 2\beta_{x3}\cos 2\tau\right)x^3 - \gamma y' = 0 \\ y'' + \alpha_y y' + \delta_y y + \delta_{y3}y^3 + \gamma x' = 0 \end{cases} \tag{5}$$

The derivative operator and the scaled parameters used in Eq. 5 are defined in Table 1.

Table 1: the derivative operator and the scaled parameters

$\alpha_x = \dfrac{2c_x}{m\omega}$	$\alpha_y = \dfrac{2c_y}{m\omega}$
$\delta_x = \dfrac{4\left(k_{x1} + r_1V_A^2\right)}{m\omega^2}$	$\delta_y = \dfrac{4k_{y1}}{m\omega^2}$
$\delta_{x3} = \dfrac{4\left(k_{x3} + r_3V_A^2\right)}{m\omega^2}$	$\delta_{y3} = \dfrac{4k_{y3}}{m\omega^2}$
$\beta_x = \dfrac{2r_1V_A^2}{m\omega^2}$	$\beta_{x3} = \dfrac{2r_3V_A^2}{m\omega^2}$
$\gamma = \dfrac{4\Omega_z}{\omega}$	$2\tau = \omega t$
$(.)' = \dfrac{d(.)}{d\tau}$	

Ruling out the nonlinear terms in Eq. 5 by setting k_3 and r_3 equal to zero, the equation of motion for the linear parametrically excited system becomes

$$\begin{cases} x'' + \alpha_x x' + \left(\delta_x + 2\beta_x\cos 2\tau\right)x - \gamma y' = 0 \\ y'' + \alpha_y y' + \delta_y y + \gamma x' = 0 \end{cases} \tag{6}$$

In this paper we study the existence and stability of periodic solutions for Eq. 5 and Eq. 6 and show that Eq. 6 has no steady periodic solution other than zero while Eq. 5 confronts some critical problems. However, before doing so we need to introduce Floquet Theory which represents a linear stability analysis of periodic systems.

FLOQUET THEORY[1]

Consider the general set of inhomogeneous, non-autonomous differential equations:

$$\dot{x} = F(x,t) = F(x,t+T) \quad where \quad F(0,t) \neq 0 \tag{7}$$

Assume that this set of differential equations has a periodic solution $x_r(t) = x_r(t+T)$. For a sufficiently small perturbation $\tilde{x}(t)$ such that we have $x(t) = x_r(t) + \tilde{x}(t)$, the dynamics of the perturbation term will be of the form

$$\dot{\tilde{x}}(t) = D(t)\tilde{x}(t) \tag{8}$$

where

$$D(t) = \left.\frac{\partial F(x,t)}{\partial x}\right|_{\tilde{x}(t)=0} = \left.\frac{\partial F(x,t)}{\partial x}\right|_{x(t)=x_r(t)} \tag{9}$$

The periodic solution of Eq. 7 is stable (i.e. the influence of perturbation $\tilde{x}(t)$ will fade away) if and only if the magnitudes of all the eigenvalues of the so-called *monodromy matrix* are less than unity. Monodromy matrix as defined in Eq. 11 is the solution of the set of differential equations Eq. 10 for the *state transition matrix* $\Phi(t)$ with identity initial condition, evaluated at the time T.

$$\dot{\Phi}(t) = D(t)\Phi(t) \quad with\ initial\ condition: \Phi(0) = I \tag{10}$$

$$M = \Phi(T) \tag{11}$$

Each eigenvalue of the monodromy matrix is referred to as a *Floquet multiplier* of the system and is a measure of the fading rate of perturbation, in the direction of the corresponding eigenvector. When the magnitudes of all Floquet multipliers are less than unity, perturbation has faded away in each direction after one period T. Because of the periodic nature of Eq. 7, it can be concluded that the perturbation $\tilde{x}(t)$ will finally approach the zero state.

If only one of the Floquet multipliers has a magnitude greater than unity, perturbation will grow exponentially in that direction and consequently $x_r(t)$, the periodic solution of the system, is unstable. When the stability properties of the system change while changing parameter values, a type of bifurcation occurs in the system which contributes to remarkable changes in its dynamical behavior.

[1] To see the proof of this theorem, one can refer to [12]Argyris, J., Faust, G., and Haase, M., 1994, *An Exploration of Chaos* vol. VII: North-Holland. pp159-169

EXPONENTIAL STABILITY OF THE LINEAR SYSTEM

In this section, we show that when the nonlinear terms of Eq. 5 are neglected and the linearized system is considered, the system can never posses any stable periodic solution other than zero, and when stability conditions predict a stable periodic response (i.e. Floquet multipliers are less than unity), this response would be the trivial solution. That's because in this case, origin is globally exponentially stable and therefore, all trajectories move towards origin. Theorem 1 shows this fact.

Theorem 1: The system in Eq. 6 has no asymptotically stable periodic response other than trivial solution. Furthermore, if the asymptotic stability conditions hold, the origin is globally exponentially stable.

Proof of Theorem 1: By defining state variables as $x_1 \triangleq x$, $x_2 \triangleq x'$, $x_3 \triangleq y$, $x_4 \triangleq y'$, the system in Eq. 6 can be rewritten in the form of Eq. 12

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(\delta_x + 2\beta_x \cos 2\tau) & -\alpha_x & 0 & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & -\gamma & -\delta_y & -\alpha_y \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{12}$$

$$or \quad \underline{x}' = A_{(\tau)}\underline{x}$$

According to Floquet Theory, the dynamics of perturbation will be of the form:

$$\underline{\tilde{x}}' = \left.\frac{\partial F}{\partial x}\right|_{x=x_r} \underline{\tilde{x}} \quad \Rightarrow$$

$$\begin{bmatrix} \tilde{x}_1' \\ \tilde{x}_2' \\ \tilde{x}_3' \\ \tilde{x}_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(\delta_x + 2\beta_x \cos 2\tau) & -\alpha_x & 0 & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & -\gamma & -\delta_y & -\alpha_y \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} \tag{13}$$

$$or \quad \underline{\tilde{x}}' = D_{(\tau)}\underline{\tilde{x}} = A_{(\tau)}\underline{\tilde{x}}$$

It can be seen that dynamics of perturbation (Eq. 13) is the same as dynamics of the system itself (Eq. 12). Consequently, if the periodic solution of Eq. 6 is asymptotically stable, the perturbation term $\tilde{x}(t)$ should move toward zero with the dynamics represented in Eq. 13; or equivalently, according to Floquet Theory, all Floquet exponents should have a magnitude less than 1. However, fading of perturbation with any initial condition means that the same will happen to the response of the system in Eq. 6 and Eq. 12 because of the same governing equations. Consequently, three cases may happen to Floquet multipliers (i.e. eigenvalues of the monodromy matrix) and accordingly to the solution of Eq. 6:

- $|\lambda_i| < 1 \quad (\forall i) \quad \Rightarrow \begin{cases} x(t) \to 0 \\ \tilde{x}(t) \to 0 \\ x_r(t) = 0 : \ \ Stable \end{cases}$

- $|\lambda_i| \le 1 \quad (\forall i)$ $\left(\exists |\lambda_k| = 1\right)$ $\Rightarrow \begin{cases} x(t) \to \text{No Specific Limit Set} \\ \tilde{x}(t) \nrightarrow 0 \\ x_r(t) \ne 0: \ \text{Quasi} - \text{Stable} \end{cases}$
- $|\lambda_i| > 1 \quad (\exists i) \Rightarrow \begin{cases} x(t) \to \infty \\ \tilde{x}(t) \to \infty \\ x_r(t) = \infty: \ \text{Unstable} \end{cases}$

Due to the linear nature of Eq. 6 and Eq. 12, this holds for any initial condition in $\mathbb{R}^2$, so we can conclude that if the stability conditions hold (based on values of the parameters of the system) the origin is globally asymptotically stable. To prove the global exponential stability, note that if the system is stable then $\forall i: \ |\lambda_i| < 1$. By denoting $\max\{|\lambda_i|\}$ by ρ we can write:

$$\left\|x_{(t+T)}\right\| = \left\|\Phi(T)x_{(t)}\right\| \le \rho \left\|x_{(t)}\right\| \tag{14}$$

Because of periodic nature of the governing equations, it can be shown that (see [13]):

$$\Phi(kT) = \Phi^k(T) \tag{15}$$

and

$$\lambda\left(\Phi(kT)\right) = \lambda\left(\Phi^k(T)\right) = \lambda^k\left(\Phi(T)\right) \tag{16}$$

So, we may write:

$$\left\|x_{(t+nT)}\right\| \le \rho^n \left\|x_{(t)}\right\| = e^{n \ln \rho} \left\|x_{(t)}\right\| \tag{17}$$

By defining $\mu \triangleq -\ln \rho$ and remembering that $0 < \rho < 1 \ \Rightarrow \ \mu > 0$ we can write:

$$\left\|x_{(t+nT)}\right\| \le e^{-n\mu} \left\|x_{(t)}\right\| \tag{18}$$

Hence, the origin shows a kind of exponential stability behavior.

STABILITY OF THE NONLINEAR SYSTEM

In this section we are going to examine whether the nonlinear system in Eq. 5 behave the same as the linear system. To this end, we first introduce a very useful theorem which relates stability of stationary point of a nonlinear system to that of a linear system. Then, we employ this theorem to show that origin is again exponentially stable for the nonlinear system, but this time the stability can only be proved locally. From the results of Theorem 3, we can conclude that having a stable periodic solution for the system in Eq. 5 is hardly possible (if not impossible) and if it exists, starting from equilibrium state as the initial condition, the system faces some difficulties reaching the periodic orbit because of exponential stability of the origin.

Theorem 2[2]: Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = F(x,t) \tag{19}$$

where $F: [0,\infty) \times D \to \mathbb{R}^n$ is continuously differentiable, $D = \left\{x \in \mathbb{R}^n \ \textit{such that} \ \|x\|_2 < r\right\}$ and the Jacobian matrix $\left[\partial F / \partial x\right]$ is bounded and Lipschitz on *D* uniformly in *t* and let:

$$A(t) = \left.\frac{\partial F(x,t)}{\partial x}\right|_{x=0} \tag{20}$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system (Eq. 19) if it is an exponentially stable equilibrium point for the linear system in Eq. 21

$$\dot{x} = A(t)x \tag{21}$$

The region of attraction for Eq. 19 contains Ω_ρ where $\Omega_\rho = \left\{\|x\|_2 \le \rho\right\}$ for some $0 < \rho \le r$.

Theorem 3: When there exists an asymptotically stable periodic orbit for the system in Eq. 5, origin is locally exponentially stable.

Proof of Theorem 3: Writing Eq. 5 in the form of Eq. 19 we have:

$$F(x,\tau) = \begin{Bmatrix} x_2 \\ -\alpha_x x_2 - \left(\delta_x + 2\beta_x \cos 2\tau\right)x_1 \\ -\left(\delta_{x3} + 2\beta_{x3}\cos 2\tau\right)x_1^3 + \gamma x_4 \\ x_4 \\ -\alpha_y x_4 - \delta_y x_3 - \delta_{y3} x_3^3 - \gamma x_2 \end{Bmatrix} \tag{22}$$

It is obvious that origin is an equilibrium point for $x' = F(x,t)$. Also it can easily be shown that *F* is continuously differentiable. The Jacobian matrix is computed as:

$$\frac{\partial F}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \begin{matrix} -\left(\delta_x + 2\beta_x \cos 2\tau\right) \\ -3\left(\delta_{x3} + 2\beta_{x3}\cos 2\tau\right)x_1^2 \end{matrix} & -\alpha_x & 0 & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & -\gamma & -\delta_y - 3\delta_{y3}x_3^2 & -\alpha_y \end{bmatrix} \tag{23}$$

which is bounded in any bounded region $[0,\infty) \times D$ where $D = \left\{x \in \mathbb{R}^n \ \textit{such that} \ \|x\|_2 < r\right\}$. The derivative of the Jacobian

[2] To find this theorem and its proof, see [13] Khalil, H. K., 2002, *Nonlinear systems*, 3rd ed.: Prentice Hall: New Jersey. p161

matrix with respect to x, which is a 3-dimensional matrix with 3×3×3 dimension, is also bounded in $\left[0,\infty\right)\times D$ so $\left[\partial F / \partial x\right]$ is Lipschitz on D. Because $\cos 2\tau$ is bounded, this Lipschitz property is uniform in t.

Also for this system, the associated linear system (as in Eq. 21) is the same as Eq. 12 for which we proved in Theorem 1 that origin is exponentially stable[3]. So all the requirements in Theorem 2 are satisfied and hence, the origin is locally exponentially stable with some region of attraction Ω containing Ω_ρ.

Discussion on the nonlinear system: Theorem 3 ensures that near origin, for any initial condition in Ω, trajectories approach the zero state. We cannot say anything outside Ω so, there might (or might not) be some stable periodic solutions in that region. But still when there is a stable periodic orbit[4] outside Ω, there remains a challenging problem which is the fact that the ability of the gyro to read the external rotation depends on its ability to leave the exponentially stable origin (rest condition) to the periodic orbit outside Ω. Consequently, if for some parameter values, there exists a stable periodic response, reaching this orbit faces some fundamental difficulties. This is why we suggest using parametro-harmonic excitation instead of parametric excitation.

PARAMETRO-HARMONIC EXCITATION

In this section, we offer a way out of the problem associated with parametric resonance and call this new method, *parametro-harmonic excitation*. We show that in the new system for some parameter values[5], periodic orbits are stable, i.e. Floquet Exponents are less than one. But before doing so, we need to introduce parametro-harmonic excitation and derive the new equation of motion for the system.

Equation of Motion: The problem with parametric excitation originated from the fact that zero was always an equilibrium point of the system. To overcome this problem, we suggest adding a constant term to Eq. 4 i.e.

$$r\left(x\right)=r_0+r_1x+r_3x^3 \qquad \textbf{(24)}$$

Replacing Eq. 24 instead of Eq. 4 in Eq. 1 and using the same re-scaling parameters as in Table 1 we will have:

$$\begin{cases} x''+\alpha_x x'+\left(\delta_x+2\beta_x\cos 2\tau\right)x+\left(\delta_{x3}+2\beta_{x3}\cos 2\tau\right)x^3 \\ \qquad -\gamma y'+\beta_0\left(1+\cos 2\tau\right)=0 \\ y''+\alpha_y y'+\delta_y y+\delta_{y3}y^3+\gamma x'=0 \end{cases} \qquad \textbf{(25)}$$

where $\beta_0=\dfrac{2r_0V_A^2}{m\omega^2}$.

Because inserting the term r_0 to the excitation function adds a harmonic term to the equation of motion while the term $r_1x_1+r_3x_1^3$ impose a parametric excitation to the system, we refer to this new kind of actuation as *parametro-harmonic excitation*.

Neglecting the nonlinear terms, the same as what we did for Eq. 6, the equation of motion for the linear parametro-harmonically excited system becomes

$$\begin{cases} x''+\alpha_x x'+\left(\delta_x+2\beta_x\cos 2\tau\right)x-\gamma y'+\beta_0\left(1+\cos 2\tau\right)=0 \\ y''+\alpha_y y'+\delta_y y+\gamma x'=0 \end{cases} \qquad \textbf{(26)}$$

Periodic Orbits: It can easily be observed that the origin is not anymore an equilibrium point for systems in Eq. 25 and Eq. 26. As a matter of fact, neither of these systems have any equilibrium point because the equilibrium conditions need x', x'', y' and y'' be zero which means that Eq. 27 for nonlinear system and Eq. 28 for linear system must be satisfied.

$$\left(\delta_x+2\beta_x\cos 2\tau\right)x+\left(\delta_{x3}+2\beta_{x3}\cos 2\tau\right)x^3=-\beta_0\left(1+\cos 2\tau\right) \qquad \textbf{(27)}$$

$$\left(\delta_x+2\beta_x\cos 2\tau\right)x=-\beta_0\left(1+\cos 2\tau\right) \qquad \textbf{(28)}$$

Except for the case where $\delta_x=2\beta_x$ & $\delta_{x3}=2\beta_{x3}$, the x found from Eq. 27 and Eq. 28 is time-varying ruling out a stationary point. However, the above condition is not possible because from Table 1 we have:

$$\begin{aligned} \delta_x &= \frac{4k_{x1}}{m\omega^2}+\frac{4r_1V_A^2}{m\omega^2}=\frac{4k_{x1}}{m\omega^2}+2\beta_x \\ \delta_{x3} &= \frac{4k_{x3}}{m\omega^2}+\frac{4r_3V_A^2}{m\omega^2}=\frac{4k_{x3}}{m\omega^2}+2\beta_{x3} \end{aligned} \qquad \textbf{(29)}$$

As a consequence, the systems in Eq. 25 and Eq. 26 have no equilibrium point. To examine the stability of periodic responses, we employ Floquet Theory. The linear system in Eq. 26 can be represented by:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\delta_x+2\beta_x\cos 2\tau\right) & -\alpha_x & 0 & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & -\gamma & -\delta_y & -\alpha_y \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}+\begin{bmatrix} 0 \\ -\beta_0 \\ 0 \\ 0 \end{bmatrix}\left(1+\cos 2\tau\right) \qquad \textbf{(30)}$$

$$or \quad \underline{x}'=F\left(x,\tau\right)=A_{(\tau)}\underline{x}+B\left(1+\cos 2\tau\right)$$

For this system, the Jacobian matrix can be written in the form:

$$\frac{\partial F}{\partial x}=\frac{\partial}{\partial x}\left(A_{(\tau)}x+B\left(1+\cos 2\tau\right)\right)=A_{(\tau)} \qquad \textbf{(31)}$$

$D(\tau)$ for the dynamics of the perturbation can be found as:

$$D_{(\tau)}=\left.\frac{\partial F}{\partial x}\right|_{x(t)=x_t(t)}=\left.A_{(\tau)}\right|_{x(t)=x_t(t)}=A_{(\tau)} \qquad \textbf{(32)}$$

[3] Note that we are only interested in the case when periodic orbits are stable which requires Floquet multipliers to be less than 1.
[4] which means all Floquet multipliers are less than unity
[5] see the section *Numerical Results* following this section

It can be seen that for the linear parametro-harmonically excited system, the Jacobian matrix does not depend on the equation of periodic orbit, so its stability analysis can be carried out without the need to find the orbit itself.

Note that from Eq. 32, the dynamics of perturbation becomes the same as Eq. 13. This means that in the same regions for which the system in Eq. 6 (and Eq. 12) has the origin as its global exponential stable equilibrium point, the system in Eq. 26 has a stable periodic orbit and the perturbations added to this periodic response will fade away exponentially. Remember that the system in Eq. 26 has no equilibrium point so we are sure that it doesn't move toward any stationary point; so it will approach its stable periodic orbit.

For the system in Eq. 25 the function $F(x,\tau)$ is presented as:

$$x' = F(x,\tau) = \left\{\begin{matrix} x_2 \\ -\alpha_x x_2 - (\delta_x + 2\beta_x \cos 2\tau) x_1 \\ -(\delta_{x3} + 2\beta_{x3}\cos 2\tau) x_1^3 + \gamma x_4 \\ -\beta_0 (1+\cos 2\tau) \\ x_4 \\ -\alpha_y x_4 - \delta_y x_3 - \delta_{y3} x_3^3 - \gamma x_2 \end{matrix}\right\} \tag{33}$$

And $D(\tau)$ for the dynamics of the perturbation is:

$$D_{(\tau)} = \left.\frac{\partial F}{\partial x}\right|_{x(t)=x_t(t)} = \left.\begin{bmatrix} 0 & 1 & 0 & 0 \\ \begin{matrix} -(\delta_x + 2\beta_x \cos 2\tau) \\ -3(\delta_{x3} + 2\beta_{x3}\cos 2\tau) x_1^2 \end{matrix} & -\alpha_x & 0 & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & -\gamma & -\delta_y - 3\delta_{y3} x_3^2 & -\alpha_y \end{bmatrix}\right|_{x(t)=x_t(t)} \tag{34}$$

Because of the dependence of dynamics of perturbation (and consequently the monodromy matrix) to the periodic orbit, applying Floquet Theory to the nonlinear system requires one to find the periodic response by some methods like harmonic balance which in turn needs the values of parameters of the system to be known. Since our goal, which was providing some stable periodic orbit for the gyroscopic system, is met by linear parametro-harmonic excitation; from this point, we only consider this case and discuss the effect of individual parameters on its stability properties.

By adding a harmonic term to parametric excitation, the main problems associated with this excitation are overcome: First, parametro-harmonically excited MEMS gyroscope has a periodic response whether the system is linear or nonlinear while the linear parametrically excited system could not have periodic orbits. Furthermore, while parametric excitation contributed to stability of origin and problems leaving the rest condition, parametro-harmonic system approaches its periodic orbit from zero state without any problem because origin is not its equilibrium point anymore.

NUMERICAL RESULTS

As stated before, the magnitudes of eigenvalues of the monodromy matrix determine stability of the periodic orbit of the system. So, one can find it of great value to know the effect of changing parameters on this stability characteristic. In the linear parametro-harmonically excited system, there are 6 different parameters that can influence the monodromy matrix which are α_x , α_y , δ_x , δ_y , β_x and γ. By assuming equal damping in both directions, which is physically reasonable, this number reduces to 5. However, because of the assumption made in Eq. 2, physical stiffness in the two directions are equal which makes , δ_x and δ_y somehow dependant. Note that according to Table 1 and as we showed in Eq. 29, we have $\delta_x = 4k_{x1}/m\omega^2 + 2\beta_x$ so the assumption $k_{x1} = k_{y1} = k$ makes it $\delta_x = \delta_y + 2\beta_x$. As a result, there remains only 4 independent variables which can better be described by k instead of δ_x and δ_y. Consequently, we study the effect of 4 independent parameters α , k , β_x and γ. As stated before, the term α represents the scaled damping in the system, k is stiffness, β_x is a scaled value of excitation voltage and γ is related to external rotation which is supposed to be measured by the gyroscope.

The nominal values for m, k_1, r_1 and Ω_z are chosen the same as the respective parameters considered by Oropeza-Ramos, et. al in [1-2]. These parameters are presented in Table 2. Excitation frequency ω is typically given two values of 100rad/sec and 10^5rad/sec, the former in the non-resonant region and the latter in the region of parametric resonance.

Table 2: nominal values considered in stability analysis

$m = 0.14\times10^{-7}\,kg$
$k_1 = 58\dfrac{\mu N}{\mu m} = 58\dfrac{N}{m}$
$r_1 = 3.9\times10^{-4}\dfrac{\mu N}{\mu m\cdot V^2} = 3.9\times10^{-4}\dfrac{N}{m\cdot V^2}$
$\Omega_z = 3.5\dfrac{rad}{s}\left(\simeq 200\dfrac{\deg}{s}\right)$
$\omega = 100\dfrac{rad}{s}$ & $\omega = 10^5\dfrac{rad}{s}$

Figures 2 through 7 are plotted for ω=100rad/sec (a non-resonant driving frequency) while figures 8 through 11 are related to ω=10^5rad/sec (a parametric resonant driving frequency). Note that Floquet exponents present the fading rate of perturbations: The smaller the Floquet multipliers the faster the fading rate of perturbation terms. In Fig. 2 maximum magnitude of Floquet multipliers for the nominal values in Table 2 and different values of α and β_x is shown. It can easily be observed that the effect of β_x is negligible while α has a stabilizing effect on the system. Note that everywhere in the domain considered, the system is stable because maximum Floquet Exponent is less than unity. The same happens when keeping β_x = 1 and changing γ (or equivalently Ω_z) as shown in Fig. 3. These two figures suggest that β_x and γ are of less importance in the stability of the periodic solutions which can also be concluded from Fig. 4.

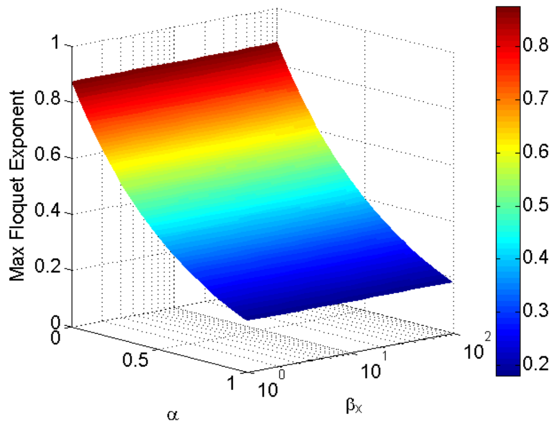

Fig. 2: MAXIMUM FLOQUET EXPONENT FOR DIFFERENT VALUES OF α AND β_x AT DRIVING FREQUENCY $\omega = 100$ *rad/s*

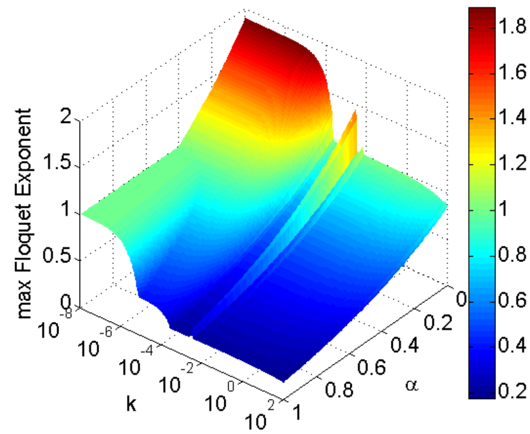

Fig. 5: MAXIMUM FLOQUET EXPONENT FOR DIFFERENT VALUES OF k AND α AT DRIVING FREQUENCY $\omega = 100$ *rad/s* WHILE KEEPING $V_a = 30v$ & $\Omega_z = 3.5$ *rad/s* (=> $\beta_x \approx 5000$ & $\gamma = 0.14$)

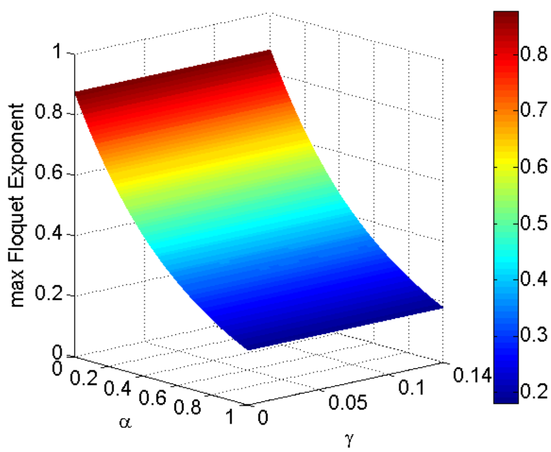

Fig. 3: MAXIMUM FLOQUET EXPONENT FOR DIFFERENT VALUES OF α AND γ AT DRIVING FREQUENCY $\omega = 100$ *rad/s*

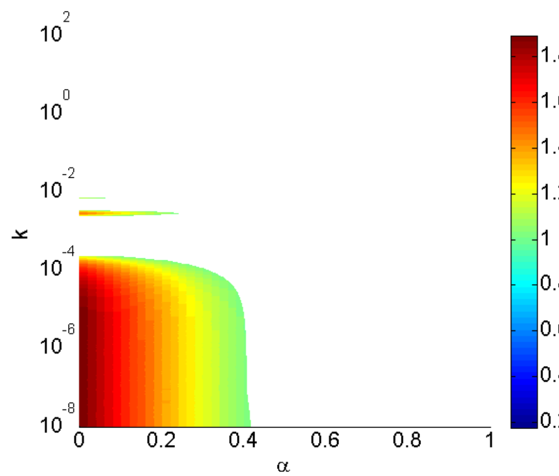

Fig. 6: REGIONS OF INSTABILITY EXTRACTED FROM Fig. 5

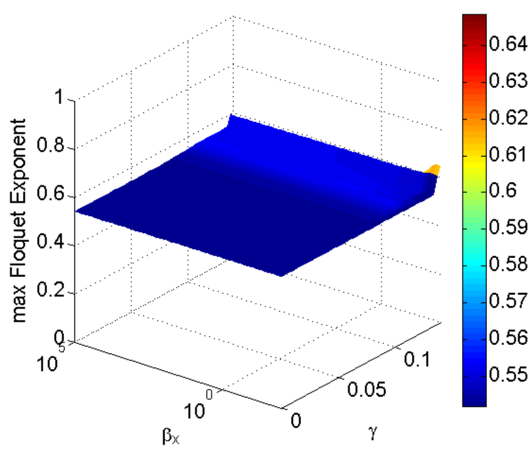

Fig. 4: MAXIMUM FLOQUET EXPONENT FOR DIFFERENT VALUES OF β_x AND γ AT $\alpha = 0.1$ AND AT DRIVING FREQUENCY $\omega = 100$ *rad/s*

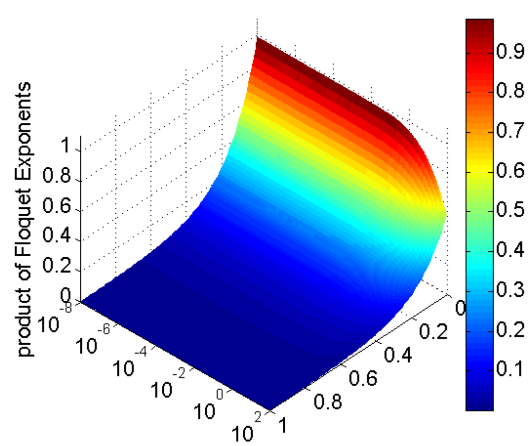

Fig. 7: PRODUCT OF FLOQUET EXPONENTS FOR DIFFERENT VALUES OF k AND α FOR $V_a = 30v$ & $\Omega_z = 3.5$ *rad/s* AT DRIVING FREQUENCY $\omega = 100$ *rad/s*

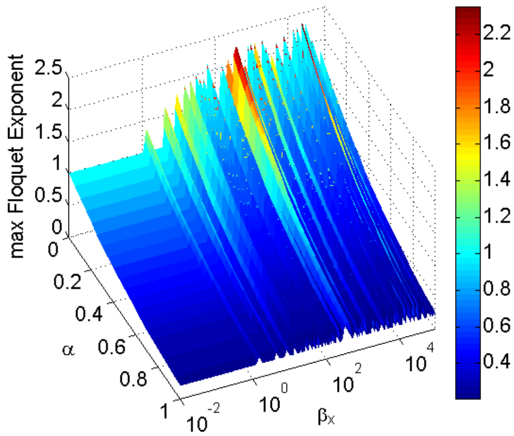

Fig. 8: MAXIMUM FLOQUET EXPONENT FOR DIFFERENT VALUES OF α AND β_x AT DRIVING FREQUENCY $\omega = 10^5$ *rad/s*

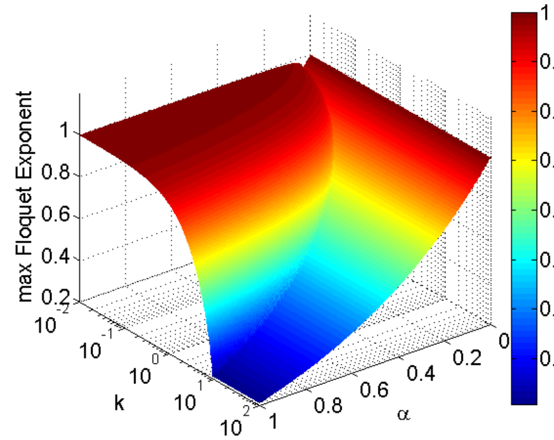

Fig. 10: MAXIMUM FLOQUET EXPONENT FOR DIFFERENT VALUES OF k AND α AT DRIVING FREQUENCY $\omega = 10^5$ *rad/s* WHILE KEEPING $V_a = 30v$ & $\Omega_z = 3.5$ *rad/s* (=> $\beta_x \approx 5000$ & $\gamma = 0.14$)

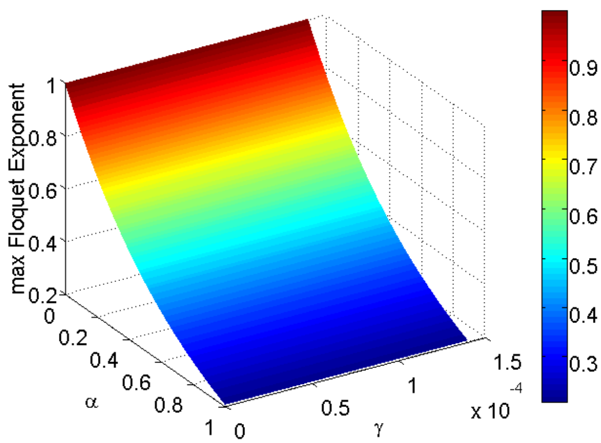

Fig. 9: MAXIMUM FLOQUET EXPONENT FOR DIFFERENT VALUES OF α AND γ AT DRIVING FREQUENCY $\omega = 10^5$ *rad/s*

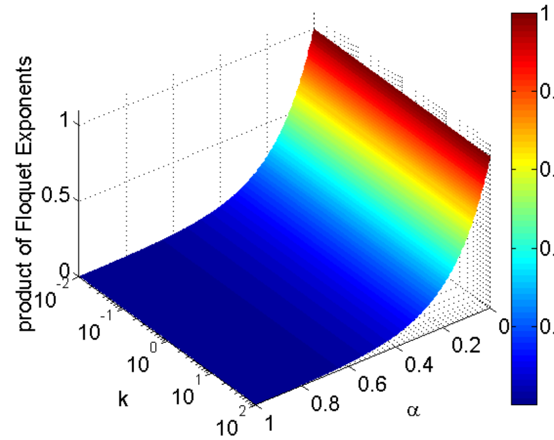

Fig. 11: PRODUCT OF FLOQUET EXPONENTS FOR DIFFERENT VALUES OF k AND α FOR $V_a = 30v$ & $\Omega_z = 3.5$ *rad/s* AT DRIVING FREQUENCY $\omega = 10^5$ *rad/s*

For all the values considered in figures 2 through 4, the periodic response is always stable. However, when changing the stiffness from its nominal value in Table 2, stability is influenced to a large extent (Fig. 5). It can be understood that increasing stiffness and damping, improves stability of the system while decreasing them weakens it until in some region, instability happens. The region of instability of Fig. 5 which is the space above the $z = 1$ plane is shown in Fig. 6. In the instability regions shown in this figure, perturbation term will grow exponentially at least in one direction in the state space. To see whether the total behavior of perturbation is in the form of expansion or contraction, we need to find the product of eigenvalues of the monodromy matrix which is presented in Fig. 7. According to this figure, everywhere in the considered domain and even in unstable regions, product of Floquet exponents is less than unity which means that perturbation has a reduction characteristic. Figures 2 through 7 suggest that in non-resonant regions only stiffness and damping play important roles in the stability of periodic response and other terms like excitation voltage and imposed external rotation are of less influence.

Behavior of the system in the parametric resonance region is rather different especially for β_x. While the effect of α and γ in resonant case (Fig. 9) is the same as respective non-resonant case (Fig. 3), there are several changes in stability behavior in Fig. 8 due to changes of β_x. Discussion on this behavior is beyond the scope of this paper. We just mention that this behavior is typical for Mathieu equation and results in *Mathieu Instability Regions* that can be observed in Fig. 8.

In Fig.10 the effect of stiffness and damping is studied. It is obvious that unlike the non-resonant excitation, for this parametric resonant excitation frequency, decreasing the stiffness does not result in instability. Contraction behavior of perturbation still holds for this case as shown in Fig. 11.

CONCLUDING REMARKS

As proved in this paper, for a MEMS gyroscope with linear stiffness and linear parametric excitation, origin has a global exponential stability and hence, all trajectories will move toward origin and not to a periodic orbit. Also for a system having nonlinear stiffness and/or nonlinear parametric excitation, it is proved that exponential stability of origin holds at least locally. This fact leads one to argue that even if there would be an asymptotically stable periodic orbit, the system has difficulties reaching it from the rest conditions because of the local exponential stability of origin.

To overcome the mentioned problem, as suggested in this paper, a new term in the excitation function is introduced which adds a harmonic term to the parametric excitation. This new actuation which is called *parametro-harmonic excitation* has the benefit that when there is an asymptotically stable periodic orbit, the system will approach that orbit and not some constant state because the new system has no equilibrium point. By using parametro-harmonic excitation, the mentioned problems which were due to parametric excitation are overcome: First, a simpler system with only linear terms can be used in design and analysis because unlike the linear parametrically excited MEMS gyroscope, the linear parametro-harmonically excited system is able to have stable periodic orbits. In addition, the problem of exponential stability of origin is overcome by making the origin a non-stationary point.

When stability of periodic responses of a linear parametro-harmonically excited MEMS gyroscope is analyzed using Floquet Theory, for a non-resonant driving frequency, changing excitation voltage and imposed external rotation has no significant effect on this stability, but stiffness and damping can influence stability to a large extent. However, for a parametric resonant excitation frequency, only external rotation rate has a null effect and stiffness, damping, and excitation voltage affect the stability of periodic orbits. Generally speaking, stability of periodic orbit is increased by increasing stiffness and damping. For a driving frequency in parametric resonance regions, the so-called Mathieu Instability Regions are observed while changing the excitation voltage.

It should be noted that the stability issue is just one of the important criteria that should be considered in the design process. There are other factors that are also important for performance of a gyroscope like the Resolution needed, required Scale Factor, desired Time-Constant, maximum Permissible Drift and maximum allowable Zero Rate Output (ZRO) that have not been analyzed in this paper.

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