# A Convex Duality Approach to Optimal Control of Killed Markov Processes

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***Abstract*— The article studies a general class of optimal control problems for stochastic systems governed by controlled Itô differential equations on compact state spaces where the process is killed (i.e. switched to zero dynamics) upon arrival on the boundary of the domain. Using the notion of occupation measures, it is shown that the original problem is embedded in a convex linear program on the space of Radon measures and, since the embedding is tight, the optimal solution of both the original and the convex relaxation problems are equal. By exploiting the dual relationship between the space of continuous functions and of measures, the value function is identified as the upper envelope of the smooth sub-solutions of the Hamilton-Jacobi problem. Using the denseness of polynomials on compact domains and employing Putinar's Positivstellensatz, a numerical algorithm is formulated for fast approximation of the value function. Examples are provided to illustrate the methodology.**

## I. Introduction

The convex duality method for optimal control problems was initiated by Vinter and Lewis [1], [2] for deterministic control systems and, later, by Fleming and Vermes for piecewise deterministic [3] and stochastic [4] processes. The fundamental idea of this approach is the introduction of a weak formulation that embeds the original (strong) problem into a convex linear program over the space of Radon measures. Upon establishing the equivalence of the two problems, new necessary and sufficient optimality condition are obtained by invoking the Fenchel-Rockafellar duality theorem. As a result, one can identify the value function of the optimal control as the upper-envelope of the set of all smooth (sub-)solutions of the Hamilton-Jacobi problem [4]. This approach has been mostly employed for the existence, uniqueness and characterization of optimal policies in some desirable classes of controls by investigating the extreme points of the set of sub-solutions (see e.g. [5]–[8]). However, it is not directly amenable to numerical investigations due to excessive numerical demands of having access to a large enough set of test functions representing the sub-solutions.

For systems defined over compact domains, however, the class of polynomial functions is a rich family to represent smooth sub-solutions due the facts that their algebra is dense in the space of continuous functions by Stone–Weierstrass theorem. Moreover, for optimal control problems with polynomial dynamics and costs, the partial differential Hamilton-Jacobi-Bellman (HJB) inequalities can be reduced to algebraic relations since, with polynomial coefficients, the associated infinitesimal generator maps polynomials into polynomials. Moreover, methodologies based upon sum of squares (SOS) techniques and Lasserre's hierarchy of finite-dimensional semi-definite programs (SDP) [9] offer polynomial-time algorithms for optimization over nonnegative polynomials.

For deterministic control systems, SOS-based numerical algorithms are established in [10]–[12] for classical optimal control problems, and in [13]–[15] for hybrid optimal control problems. For stochastic optimal control problems, however, polynomial-based numerical algorithms have been subject to a limited number of studies. safety analysis (see e.g. [16]). Within this context, stochastic optimal control discussions (e.g. in [17]) do not directly invoke the dual relationship between the space of continuous functions and of measures.

The first objective of this paper is the representation of the convex duality method for finite horizon stochastic optimal control problems for killed Markov processes that are defined over compact subsets of Euclidean state spaces. Closely related, but fundamentally distinct frameworks, have been presented for exit-time problems with bounded state spaces and infinite time horizons (see e.g. [18], [19, Chapter V]), and for optimal stopping problems with unbounded state spaces and finite horizons (see e.g. [8]). To this end, this article presents modification of the framework of Fleming and Vermes [4] in the following aspects. First, the discussion in [4] centers around infinite horizon problems on compactified (hence, unbounded) domains[1]. Furthermore, their framework precludes discussion of polynomial dynamics by requiring linear growth of drift and diffusion to ensure existence and uniqueness of solutions on unbounded domains. Moreover, the notion of weighted spaces in [4], as seen in this paper, is unnecessary on bounded domains. Last but not least, the inclusion of leaving cost as well as terminal cost, that appear only implicitly in [4], is an indispensable part of the discussion of the current paper. The second objective of this work is the presentation of SOS-based techniques and Lasserre's hierarchy of finite-dimensional SDPs as an alternative methodology for the solution of stochastic optimal control problems.

The organization of the paper is as follows. Section II presents the class of controlled Markov processes killed at the boundary and the associated optimal control problem with running, leaving and terminal costs. In Section III we introduce the notion of occupation measures as the

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[1] The state value set in [4] is taken to be $\overline{\mathbb{R}}^n$, the one-point compactification of $\mathbb{R}^n$.

conditional expectations of sample paths intersecting with Borel subsets of the input-state-time space, and we show that the total expected cost can be strongly represented as a linear functional of occupation measures. In Section IV we show that the strong problem is tightly embedded in a linear program (LP) defined on a convex domain in the space of signed measures and we invoke duality relationship between the space of measures and that of continuous functions to prove that the value function is the upper envelope of smooth sub-solutions to a set of HJB-type inequalities. In Section V we present a numerical algorithm for the subclass of problems with polynomial dynamics and costs and where the state and input value sets are basic semi-algebraic. Examples are provided in this section to illustrate the methodology. Section VI provides concluding remarks and future directions.

## II. Problem Statement

Let $(\Omega,\Im,\Im^t,P)$ be a probability space with $\Im^t$ an increasing family of sub $\sigma$-algebras of $\Im$ such that $\Im^0$ contains the $P$-null sets, and $\Im^T=\Im$ for a fixed terminal time $T$. Let $w$ be a standard Wiener process whose associated sigma-algebra generates the natural filtration $\Im^t=\sigma\{w_s : 0\le s\le t\}$.

We study nonlinear stochastic systems governed by controlled Itô differential equations of the form

$$dx_s = f(s,x_s,u_s)\,ds + g(s,x_s)\,dw_s, \tag{1}$$

where $x_s\in X\subset\mathbb{R}^n$ and $u_s\in U\subset\mathbb{R}^m$. The sets $X$ and $U$ are assumed to be convex and compact and the functions $f$ and $g$ are considered to be Lipschitz continuous functions, respectively, on $[0,T]\times X\times U$ and $[0,T]\times X$. We remark that continuity on compact domains necessarily results in boundedness of $f$ and $g$, hence together with the Lipschitz conditions guarantee the existence and uniqueness of trajectories almost surely. We denote the interior of the state value set by $X^0$ and its boundary by $X^\partial$ and hence, $X=X^0\cup X^\partial$.

A nonanticipative, $U$-valued, input processes is denoted by $\boldsymbol{u}:=\{u_s : t\le s\le T\}$ where $u_s\in U$ is progressively measurable with respect to $\Im^s$ for all $s\in[t,T]$. The set of all nonanticipative, $U$-valued controls is denoted by $\mathscr{U}$. For each $\boldsymbol{u}\in\mathscr{U}$ we denote by $\boldsymbol{x}^{\boldsymbol{u}}:=\{x_s\equiv x_s^{\boldsymbol{u}} : t\le s\le T\}$ the solution of the stochastic differential equation (1) corresponding to $\boldsymbol{u}$ and satisfying the initial condition $x_t^{\boldsymbol{u}}\equiv x_t \overset{a.s.}{=} x\in X^0$, that is $P_{t,x}^{\boldsymbol{u}}(x_t=x)=1$, where $P_{t,x}^{\boldsymbol{u}}$ denotes the conditional probability with respect to $\Im^t$ that, due to the Markovian property and the complete observation of state, is equivalent to the conditioning on $x_s=x$ at $s=t$. Upon arrival of $\boldsymbol{x}^{\boldsymbol{u}}$ on the boundary $X^\partial$, trajectories are absorbed into a killed state $x_\theta\equiv x_{\theta-}\in X^\partial$, that is, $x_s=x_\theta$ for all $s\ge\theta$, where $\theta\in[0,T]$ is the first arrival time on the boundary $X^\partial$ and $x_{\theta-}:=\lim_{s\uparrow\theta}x_s$ is the left limit of the state upon arrival on the boundary. In other words, for every sample path, either it is the case that $x_s^{\boldsymbol{u}}\in X^0$ for all $s\in[t,T]$ or there exist $\theta\in(t,T]$ such that $x_s^{\boldsymbol{u}}\in X^0$ for all $s\in[t,\theta)$ and $x_s^{\boldsymbol{u}}=x_\theta\in X^\partial$ for all $s\in[\theta,T]$[2].

[2]In the stochastic hybrid systems terminology, this is equivalent to an autonomous switching to another mode (discrete state) with zero dynamics, i.e., $dx_s=f_2(s,x_s,u_s)dt+g_2(s,x_s)dw_s\equiv 0$.

The cost-to-go associated with $\boldsymbol{u}$ is considered to be

$$J(t,x,\boldsymbol{u})=\mathbb{E}_{t,x}^{\boldsymbol{u}}\left\{\int_t^{\min\{\theta,T\}} l(x_s,u_s)\,ds + \mathbb{I}_{[t,T)}(\theta)\cdot\ell(\theta,x_\theta)+\mathbb{I}^c_{[t,T)}(\theta)\cdot L(x_T)\right\}, \tag{2}$$

where $l$, $\ell$ and $L$ are continuous functions bounded on their respective domains, and $\mathbb{E}_{t,x}^{\boldsymbol{u}}$ denotes the (conditional) expectation for trajectories passing through $x_t$ at $t$ and evolving under the input policy $\boldsymbol{u}$. For every sample path, the indicator function $\mathbb{I}_{[t,T)}(\theta)$ is equal to 1 if there exists a stopping time $\theta\in[t,T)$, and it is equal to 0 otherwise; we also use $\mathbb{I}^c_{[t,T)}(\theta):=1-\mathbb{I}_{[t,T)}(\theta)$.

The value function of optimal control is defined as

$$V(t,x):=\inf_{\boldsymbol{u}\in\mathscr{U}} J(t,x,\boldsymbol{u}). \tag{3}$$

## III. Strong Formulation

In this section, we present the optimal control problem in the space of measures. We refer to this reformulation as the "strong" problem due to the direct correspondence between [conditional] expectations in both the functional and the measure formulations in this section.

### A. Occupation Measures

We define the *input-state-time occupation measure* as

$$\mu^{\boldsymbol{u}}\left(B_t,B_x^0,B_u\right):=\mathbb{E}_{t,x}^{\boldsymbol{u}}\int_{B_t\cap[t,T)}\mathbb{I}_{B_x^0}(x_s)\cdot\mathbb{I}_{B_u}(u_s)\,ds, \tag{4}$$

for arbitrary Borel sets $B_t\subset[0,T]$, $B_x^0\subset X^0$, $B_u\subset U$, where $\mathbb{I}_B$ denotes the indicator function of the set $B$.

We also define the *absorbing state-time occupation measure* as

$$\eta^{\boldsymbol{u}}\left(B_t,B_x^\partial\right):=P_{t,x}^{\boldsymbol{u}}\left(\mathbb{I}_{[t,T)\cap B_t}(\theta)=1\ ,\ x_{\theta-}^{\boldsymbol{u}}\in B_x^\partial\right), \tag{5}$$

for arbitrary Borel sets $B_t\subset[0,T]$, $B_x^\partial\subset X^\partial$; and define the *terminal state occupation measure* as

$$\kappa^{\boldsymbol{u}}(B_x):=P_{t,x}^{\boldsymbol{u}}\left(\mathbb{I}_{[t,T)}(\theta)=0\ ,\ x_T^{\boldsymbol{u}}\in B_x\right). \tag{6}$$

for arbitrary Borel sets $B_x\subset X=X^0\cup X^\partial$.

We denote by $\mathscr{M}_S$ the set of occupations measures corresponding to all $\boldsymbol{u}\in\mathscr{U}$, i.e.,

$$\mathscr{M}_S:=\{(\mu^{\boldsymbol{u}},\eta^{\boldsymbol{u}},\kappa^{\boldsymbol{u}}) : \boldsymbol{u}\in\mathscr{U}\} \tag{7}$$

*Lemma 3.1:* If $f(s,x,U):=\{f(s,x,u) : u\in U\}$ is convex for all $s\in[0,T]$, $x\in X$ then

$$\mathscr{M}_S=\overline{\mathrm{conv}}\mathscr{M}_S \tag{8}$$

where $\overline{\mathrm{conv}}\mathscr{M}_S$ denotes the w*-convex closure of $\mathscr{M}_S$.

*Proof:* The proof is based upon the McShane–Warfield implicit function theorem (see e.g. [20]). ∎

*Lemma 3.2:* For every $\boldsymbol{u}\in\mathscr{U}$, measurable functions $l:[0,T)\times X^0\times U\to\mathbb{R}$ with $l(s,x,U):=\{l(s,x,u) : u\in U\}$

convex for all $s \in [0,T]$, $x \in X$, and for all measurable functions $\ell : [0,T) \times X^{\partial} \to \mathbb{R}$, $L : X \to \mathbb{R}$

$$\mathbb{E}_{t,x}^{\boldsymbol{u}} \int_{t}^{\min\{\theta,T\}} l\left(x_s,u_s\right) ds = \int_{[t,T)\times X^0\times U} l(x,u)\, \mu^{\boldsymbol{u}}(dt,dx,du) =: \left\langle l, \mu^{\boldsymbol{u}} \right\rangle \tag{9}$$

$$\mathbb{E}_{t,x}^{\boldsymbol{u}} \left\{ \mathbb{I}_{[t,T)}(\theta) \cdot \ell\left(\theta, x_\theta\right) \right\} = \int_{[t,T)\times X^{\partial}} \ell(t,x)\, \eta^{\boldsymbol{u}}(dt,dx) \quad := \left\langle \ell, \eta^{\boldsymbol{u}} \right\rangle \tag{10}$$

$$\mathbb{E}_{t,x}^{\boldsymbol{u}} \left\{ \mathbb{I}_{[t,T)}^{c}(\theta) \cdot L\left(x_T\right) \right\} = \int_{X} L(x)\, \kappa^{\boldsymbol{u}}(dx) \qquad := \left\langle L, \kappa^{\boldsymbol{u}} \right\rangle \tag{11}$$

*Proof:* The proof follows directly from the definitions (4), (5) and (6), and the measurability of functions $l, \ell, L$. ■

*Corollary 3.3 (Strong Problem):* Under the assumptions of Lemma 3.1 and 3.2, the value function (3) is equal to the following formulation in terms of occupation measures.

$$V(t,x) = \inf_{(\mu^{\boldsymbol{u}},\eta^{\boldsymbol{u}},\kappa^{\boldsymbol{u}})\in\mathscr{M}_S} \left\{ \left\langle l, \mu^{\boldsymbol{u}} \right\rangle + \left\langle \ell, \eta^{\boldsymbol{u}} \right\rangle + \left\langle L, \kappa^{\boldsymbol{u}} \right\rangle \right\} \tag{12}$$

### B. Infinitesimal Operator

The infinitesimal operator of the Markov process $\boldsymbol{x}$ corresponding to the constant control $\boldsymbol{u}_t \equiv u$ is defined for every twice continuously differentiable function $v \in C^2\left([0,T) \times X^0\right)$ and is given by the expression

$$\mathscr{A}^{u} v(t,x) = \frac{\partial v(t,x)}{\partial t} + \left\langle f(t,x,u), \frac{\partial v(t,x)}{\partial x} \right\rangle + \frac{1}{2} \mathrm{tr}\left( g(t,x)^T g(t,x) \frac{\partial^2 v(t,x)}{\partial x^2} \right). \tag{13}$$

We also use this notation for nonsmooth functions $v$ to denote the value of the expression on the right-hand side at every point $(t,x) \in [0,T) \times X^0$ where the corresponding partial derivatives exist.

### C. Dynkin Formula

It follows from Itô's formula that for arbitrary non-anticipative input processes $\boldsymbol{u} \in \mathscr{U}$ the generalization of the fundamental theorem of calculus (Dynkin's formula) holds true. In particular, for every sample path, for all $v \in C^2([0,T] \times X)$, and for every $\tau \in [t, \min\{\theta, T\})$ one has

$$v(\tau, x_\tau) - v(t,x) = \int_t^{\tau} \mathscr{A}^{u_s} v(s,x_s)\, ds + \int_t^{\tau} \left\langle g(s,x_s), \frac{\partial v(s,x_s)}{\partial x_s} \right\rangle dw_s \tag{14}$$

Moreover, for every killed sample path, for all $v \in C^2([0,T] \times X)$, and for every $\tau \in [\theta, T]$, one necessarily has $v(\tau, x_\tau) = v(\theta, x_\theta)$ and therefore

$$v(\tau, x_\tau) - v(t,x) = \int_t^{\theta} \mathscr{A}^{u_s} v(s,x_s)\, ds + \int_t^{\theta} \left\langle g(s,x_s), \frac{\partial v(s,x_s)}{\partial x_s} \right\rangle dw_s \tag{15}$$

If for all $v \in C^2\left([0,T) \times X^0\right)$,

$$\mathbb{E}_{t,x}^{\boldsymbol{u}} \left\{ \int_t^{\min\{\tau,\theta\}} \left[ \frac{\partial v(s,x_s)}{\partial x_s} \right]^T g(s,x_s)^T g(s,x_s) \frac{\partial v(s,x_s)}{\partial x_s} ds \right\} < \infty \tag{16}$$

then we can take expectations on both sides of (14) and (15) in order to write

$$\begin{aligned} \mathbb{E}_{t,x}^{\boldsymbol{u}} v(\tau, x_\tau) &= \mathbb{E}_{t,x}^{\boldsymbol{u}} \left\{ \mathbb{I}_{[t,T)}(\theta) \cdot v(\theta, x_\theta) + \mathbb{I}_{[t,T)}^{c}(\theta) \cdot v(T, x_T) \right\} \\ &= P_{t,x}^{\boldsymbol{u}}(\theta \in [t,T)) \left( v(t,x) + \mathbb{E}_{t,x}^{\boldsymbol{u}} \int_t^{\theta} \mathscr{A}^{u_s} v(s,x_s)\, ds \right) \\ &\quad + P_{t,x}^{\boldsymbol{u}}(\theta \notin [t,T)) \left( v(t,x) + \mathbb{E}_{t,x}^{\boldsymbol{u}} \int_t^{T} \mathscr{A}^{u_s} v(s,x_s)\, ds \right) \\ &= v(t,x) + \mathbb{E}_{t,x}^{\boldsymbol{u}} \int_t^{\min\{\theta,T\}} \mathscr{A}^{u_s} v(s,x_s)\, ds \end{aligned} \tag{17}$$

for all $\boldsymbol{u} \in \mathscr{U}$, where we have employed the fact that $P_{t,x}^{\boldsymbol{u}}\left(\mathbb{I}_{[t,T)}(\theta) = 1\right) + P_{t,x}^{\boldsymbol{u}}\left(\mathbb{I}_{[t,T)}^{c}(\theta) = 1\right) = 1$.

Since for every $v \in C^2([0,T] \times X)$, its differential form $\mathscr{A} v$ is continuous and hence measurable, we can rewrite (17) using occupation measures in the form of

$$\left\langle v, \eta^{\boldsymbol{u}} \right\rangle + \left\langle v, \kappa^{\boldsymbol{u}} \right\rangle = v(t,x) + \left\langle \mathscr{A} v, \mu^{\boldsymbol{u}} \right\rangle \tag{18}$$

### D. Adjoint Operator

We define the adjoint $\mathscr{A}^*$ of (13) as the operator satisfying

$$\left\langle \mathscr{A} v, \mu \right\rangle = \left\langle v, \mathscr{A}^* \mu \right\rangle \tag{19}$$

for every signed Borel measure $\mu \in \mathfrak{M}_{\pm}\left([0,T) \times X^0 \times U\right)$ and any twice continuously differentiable function $v \in C^2\left([0,T) \times X^0\right)$.

*Remark 3.4:* If $\mu \equiv p(s,x_s|t,x)$, where $p(s,x_s|t,x)$ is a twice continuously differentiable probability density function, then $\mathscr{A}^*$ is equivalent to the Fokker Planck operator

$$\mathscr{A}^* \mu = -\frac{\partial \mu}{\partial t} - \sum_{i=1}^{n} \frac{\partial \left( f^{(i)} \mu \right)}{\partial x^{(i)}} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \left( \left[ g^T g \right]^{(i,j)} \mu \right)}{\partial x^{(i)} \partial x^{(j)}}. \tag{20}$$

However, the derivatives of measures shall be understood in the weak sense [21] since $\mu^{\boldsymbol{u}}$ is not necessarily differentiable, nor is required to be a probability measure (due to the killing of the process upon arrival of sample paths on $X^{\partial}$). Since the concept of weak derivatives of measures is not essential for the discussion of this paper, we employ the definitional form (19) of $\mathscr{A}^*$ rather than its closed form expression in (20).

### E. Properties of Occupation Measures

Introducing the notation $\delta_{t,x}$ for the Dirac delta measure at $(t,x)$, the relation (18) can be equivalently written as

$$\left\langle v, \eta^{\boldsymbol{u}} \right\rangle + \left\langle v, \kappa^{\boldsymbol{u}} \right\rangle = \left\langle v, \delta_{t,x} \right\rangle + \left\langle v, \mathscr{A}^* \mu^{\boldsymbol{u}} \right\rangle \tag{21}$$

for all $v \in C^2([0,T] \times X)$. In other words, for every $\boldsymbol{u} \in \mathscr{U}$, occupation measures corresponding to trajectories of the system (1) satisfy

$$\eta^{\boldsymbol{u}} + \kappa^{\boldsymbol{u}} = \delta_{t,x} + \mathscr{A}^* \mu^{\boldsymbol{u}} \tag{22}$$

Moreover, defining the norm of a general signed measure as $\|M\| = \int dM^+ + \int dM^-$ with $M^+$ and $M^-$ denoting, respectively, the positive and negative pars of the Hahn–Jordan decomposition of $M$, one can easily verify that

$$\|\mu^{\boldsymbol{u}}\| \leq T - t \tag{23}$$

$$\|\eta^{\boldsymbol{u}}\| + \|\kappa^{\boldsymbol{u}}\| = 1 \tag{24}$$

It also follows directly from the definitions (4), (5) and (6) that for all $B_t \subset [t, T]$, $B_x \in X$ and $B_u \in U$, one obtains $\mu^{\boldsymbol{u}}(B_t, B_x, B_u) \geq 0$, $\eta^{\boldsymbol{u}}(B_t, B_x) \geq 0$, and $\kappa^{\boldsymbol{u}}(B_x) \geq 0$.

## IV. Weak Formulation

In Section III-A we have shown in Corollary 3.3 that the value function can be expressed as the infimum of a linear functional (12) over occupation measures associated with input policies. We have also shown in Section III-C that for any admissible input process $\boldsymbol{u} \in \mathscr{U}$ the associated occupation measures $\mu^{\boldsymbol{u}}, \eta^{\boldsymbol{u}}, \kappa^{\boldsymbol{u}}$ satisfy positivity and bounded norm conditions as well as a linear constraints (21). However, direct analysis on $\mathscr{M}_S$ is challenging due to its dependence on $\mathscr{U}$. Hence, we introduce a weaker problem define over a larger domain $\mathscr{M}_W \supset \mathscr{M}_S$ that is easily identifiable as a convex domain in the space of measures.

### A. Weak Problem

Let $\mathfrak{M}_\pm(S)$ denote the set of all signed Borel measures on $S$ and $\mathfrak{M}_+(S)$ the non-negative cone of $\mathfrak{M}_\pm(S)$. Define $M := (\mu, \eta, \kappa) \in \mathfrak{M}_\pm([0, T] \times X \times U)$ such that $\mu \in \mathfrak{M}_\pm\left([0, T) \times X^0 \times U\right)$, $\eta \in \mathfrak{M}_\pm\left([0, T) \times X^\partial\right)$, and $\kappa \in \mathfrak{M}_\pm(X)$.

For every $M \in \mathfrak{M}_\pm([0, T] \times X \times U)$ we define the norm by

$$\begin{aligned} \|M\| &\equiv \|(\mu, \eta, \kappa)\| := \|\mu\| + \|\eta\| + \|\kappa\| \\ &= \int_{[t,T)\times X^0 \times U} d\mu^+ + \int_{[t,T)\times X^0 \times U} d\mu^- + \int_{[t,T)\times X^\partial} d\eta^+ \\ &\quad + \int_{[t,T)\times X^\partial} d\eta^- + \int_X d\kappa^+ + \int_X d\kappa^- \end{aligned} \tag{25}$$

We define the weak problem and the corresponding weak value function as

$$W(t, x) := \min_{M \in \mathscr{M}_W} \langle l, \mu \rangle + \langle \ell, \eta \rangle + \langle L, \kappa \rangle \tag{26}$$

where $\mathscr{M}_W := \mathscr{M}_{PB} \cap \mathscr{M}_{\mathscr{A}}$, with

$$\mathscr{M}_{PB} := \Big\{ M \equiv (\mu, \eta, \kappa) \in \mathfrak{M}_+([0, T] \times X \times U) : \|M\| \leq T - t + 1 \Big\} \tag{27}$$

and

$$\mathscr{M}_{\mathscr{A}} := \Big\{ M \equiv (\mu, \eta, \kappa) \in \mathfrak{M}_\pm([0, T] \times X \times U) : \eta + \kappa = \delta_{t,x} + \mathscr{A}^* \mu \Big\} \tag{28}$$

The above problem is a linear program on the space of signed measures. The set $\mathscr{M}_{PB}$ is a convex subset of $\mathfrak{M}_\pm$ and the constraint $\mathscr{M}_{\mathscr{A}}$ is linear and therefore restricts the problem into a linear subspace. Endowing the space of continuous functions with the topology of the sup-norm and endowing the space of signed measures, $\mathfrak{M}_\pm$, with a weak dual topology, it follows that $\mathscr{M}_{PB} \cap \mathscr{M}_{\mathscr{A}}$ is w*-compact and hence, the infimum is achieved and is equal to the minimum.

### B. Measure – Function Duality

Over the compact Hausdorff space $[0, T] \times X \times U$, the Banach space of continuous functions $C([0, T] \times X \times U)$ equipped with the sup-norm has a topological dual $C^*([0, T] \times X \times U)$ that is isometrically isomorphic to $\mathfrak{M}_\pm([0, T] \times X \times U)$ equipped with the norm (25). The norm topology of $C([0, T] \times X \times U)$ and the weak dual topology of $\mathfrak{M}_\pm([0, T] \times X \times U)$ are compatible with the pairing defined by the bilinear form

$$\langle c, M \rangle = \langle c^0, \mu \rangle + \langle c^\partial, \eta \rangle + \langle c^T, \kappa \rangle \tag{29}$$

for all $c \equiv (c^0, c^\partial, c^T) \in C([0, T] \times X \times U)$ and $M \equiv (\mu, \eta, \kappa) \in \mathfrak{M}_\pm([0, T] \times X \times U)$.

### C. Fenchel Normal Form

Using the notion of weak value function, we reformulate the convexly constrained linear program as an unconstrained convex problem by introducing the functionals $h_1$ and $h_2$ : $\mathfrak{M}_\pm([0, T] \times X \times U) \to \overline{\mathbb{R}}$ by

$$h_1(M) := \begin{cases} \langle l, \mu \rangle + \langle \ell, \eta \rangle + \langle L, \kappa \rangle & \text{if } M \equiv (\mu, \eta, \kappa) \in \mathscr{M}_{PB} \\ +\infty & \text{otherwise} \end{cases} \tag{30}$$

$$h_2(M) := \begin{cases} 0 & \text{if } M \equiv (\mu, \eta, \kappa) \in \mathscr{M}_{\mathscr{A}} \\ -\infty & \text{otherwise} \end{cases} \tag{31}$$

Both $h_1$ and $-h_2$ are convex and lower semi-continuous [4] and, hence,

$$W(t, x) = \min_{M \in \mathfrak{M}_\pm([0,T]\times X \times U)} h_1(M) - h_2(M) \tag{32}$$

### D. Legendre-Fenchel Transform

In order to employ duality methods, we employ the Legendre-Fenchel transform, which is a generalization of the Legendre transform commonly encountered in calculus of variations and in classical mechanics. The real-valued functional $h_1$ is convex and its convex conjugate (Legendre-Fenchel transform) is defined by

$$h_1^*(c) := \sup_{M \in \mathfrak{M}_\pm([0,T]\times X \times U)} \Big\{ \langle c, M \rangle - h_1(M) \Big\} \tag{33}$$

*Lemma 4.1:*

$$\begin{aligned} h_1^*(c) &\equiv \sup_{M \in \mathscr{M}_{PB}} \Big\{ \langle c^0, \mu \rangle + \langle c^\partial, \eta \rangle + \langle c^T, \kappa \rangle \\ &\qquad - \langle l, \mu \rangle + \langle \ell, \eta \rangle + \langle L, \kappa \rangle \Big\} \\ &= (T - t + 1) \Big( \|(c^0 - l)^+\| + \|(c^\partial - \ell)^+\| + \|(c^T - L)^+\| \Big) \end{aligned} \tag{34}$$

where $(f)^+$ denotes the positive part of the function $f$, i.e., $f^+(x) = max\{0, f(x)\}$.

*Proof:* The proof is a modification of [4, Lemma 4.1] and is removed due to space limitations. ■

For the concave functional $h_2$ the Legendre-Fenchel transform is defined as

$$h_2^*(c) := \inf_{M\in\mathfrak{M}_\pm([0,T]\times X\times U)} \Big\{ \big\langle c, M \big\rangle - h_2(M) \Big\} \tag{35}$$

*Lemma 4.2:*

$$h_2^*(c) \equiv \inf_{M\in\mathscr{M}_\mathscr{A}} \Big\{ \big\langle c^0,\mu \big\rangle + \big\langle c^\partial,\eta \big\rangle + \big\langle c^T,\kappa \big\rangle \Big\}$$
$$= \begin{cases} \lim\limits_{i\to\infty} v_i(t,x) & \text{if } \begin{cases} c^0 = -\lim\limits_{i\to\infty} \mathscr{A} v_i \\ c^\partial = \lim\limits_{i\to\infty} v_i^\partial \\ c^T = \lim\limits_{i\to\infty} v_i^T \end{cases} \\ -\infty & \text{otherwise} \end{cases} \tag{36}$$

*Proof:* The proof is a modification of [4, Lemma 4.2] and is removed due to space limitations. ■

### E. The Hamilton-Jacobi Problem

Applying the Rockafellar duality theorem (see e.g. [4]) to $C^*([0,T]\times X\times U) = \mathfrak{M}_\pm([0,T]\times X\times U)$, we obtain

$$\min_{M\in\mathfrak{M}_\pm([0,T]\times X\times U)} \Big\{ h_1(M) - h_2(M) \Big\} = \sup_{c\in C([0,T]\times X\times U)} \Big\{ h_2^*(c) - h_1^*(c) \Big\} \tag{37}$$

whenever the set $\{c : h_2^*(c) > -\infty\}$ contains a continuity point of $h_1^*(c)$ that is finite. Since $h_1^*$ is continuous and finite on whole $C([0,T]\times X\times U)$ and $h_2^*$ is not identically $-\infty$ we deduce that (37) holds.

*Theorem 4.3:*

$$W(t,x) = \sup_{v\in C^2([0,T]\times X)} v(t,x) : \begin{cases} \mathscr{A} v + l \geq 0 \\ v^\partial - \ell \leq 0 \\ v^T - L \leq 0 \end{cases} \tag{38}$$

*Proof:*

$$\tilde{v} := v - (T-t+1)(\|(\mathscr{A} v + l)^-\| + \|(v^\partial - \ell)^+\| + \|(v^T - L)^+\|) \tag{39}$$

Then

$$\begin{aligned} &\mathscr{A}\tilde{v} + l \\ &\equiv \mathscr{A} v + l + \|(\mathscr{A} v + l)^-\| + \|(v^\partial - \ell)^+\| + \|(v^T - L)^+\| \\ &\geq \mathscr{A} v + l + \sup_{(s,x,u)\in[0,T]\times X\times U} |(\mathscr{A}^u v(s,x) + l(x,u))^-| \geq 0 \end{aligned} \tag{40}$$

$$\begin{aligned} \tilde{v}^\partial - \ell &\leq v^\partial - \ell - (T-t+1)\|(v^\partial - \ell)^+\| \\ &\leq v^\partial - \ell - \|(v^\partial - \ell)^+\| \leq 0 \end{aligned} \tag{41}$$

and, similarly,

$$\tilde{v}^T - L \leq 0 \tag{42}$$

■

### F. Equivalence of the Weak and Strong Problems

It follows from the definition (26) of the weak value function that

$$\begin{aligned} W(t,x) &= \min_{(\mu,\eta,\kappa)\in\mathscr{M}_{PB}\cap\mathscr{M}_\mathscr{A}} \big\langle l,\mu \big\rangle + \big\langle \ell,\eta \big\rangle + \big\langle L,\kappa \big\rangle \\ &\leq V(t,x) = \inf_{(\mu^{\boldsymbol{u}},\eta^{\boldsymbol{u}},\kappa^{\boldsymbol{u}})\in\mathscr{M}_S} \Big\{ \big\langle l,\mu^{\boldsymbol{u}} \big\rangle + \big\langle \ell,\eta^{\boldsymbol{u}} \big\rangle + \big\langle L,\kappa^{\boldsymbol{u}} \big\rangle \Big\} \end{aligned} \tag{43}$$

since $\mathscr{M}_S \subset \mathscr{M}_{PB}\cap\mathscr{M}_\mathscr{A}$. In order to show the equivalence of the weak and the strong problems, we need to show that strict inequality cannot hold and hence, the weak and the strong value functions coincide.

*Theorem 4.4:* The weak and the strong value functions are equal, i.e.,

$$W(t,x) = V(t,x) \tag{44}$$

*Proof:* Let's assume that this is not true, i.e., there exist $(\mu_0,\eta_0,\kappa_0) \in \mathscr{M}_{PB}\cap\mathscr{M}_\mathscr{A} \setminus \mathscr{M}_S$ such that

$$\begin{aligned} W_0(t,x) &= \big\langle l,\mu_0 \big\rangle + \big\langle \ell,\eta_0 \big\rangle + \big\langle L,\kappa_0 \big\rangle \\ &< V(t,x) = \inf_{(\mu^{\boldsymbol{u}},\eta^{\boldsymbol{u}},\kappa^{\boldsymbol{u}})\in\mathscr{M}_S} \Big\{ \big\langle l,\mu^{\boldsymbol{u}} \big\rangle + \big\langle \ell,\eta^{\boldsymbol{u}} \big\rangle + \big\langle L,\kappa^{\boldsymbol{u}} \big\rangle \Big\} \end{aligned} \tag{45}$$

This means that the w*-continuous linear functional $\big\langle l,\mu \big\rangle + \big\langle \ell,\eta \big\rangle + \big\langle L,\kappa \big\rangle$ separates an element $(\mu_0,\eta_0,\kappa_0) \in \mathscr{M}_{PB}\cap\mathscr{M}_\mathscr{A}$ from the w* convex closure $\overline{\mathrm{cov}\mathscr{M}_S}$ of $\mathscr{M}_S$. Then by [4, Theorem 3], for every $\varepsilon > 0$ there exists $V^{(\varepsilon)}$ whose partial derivatives $V_t^{(\varepsilon)}$, $V_{x_i}^{(\varepsilon)}$, $V_{x_ix_j}^{(\varepsilon)}$ are defined almost everywhere, are essentially bounded and, further,

$$\|V - V^{(\varepsilon)}\| \leq \varepsilon \tag{46}$$

$$\mathscr{A}^u V^{(\varepsilon)}(s,x) + l(x,u) \geq 0 \tag{47}$$

for all $(s,x,u) \in [t,T]\times X^0\times U$,

$$V^{(\varepsilon)}(s,x) - \ell(s,x) \leq 0 \tag{48}$$

for all $(s,x) \in (t,T)\times X^\partial$, and

$$V^{(\varepsilon)}(T,x) - L(x) \leq 0 \tag{49}$$

for all $x \in X$. Since $V^{(\varepsilon)}$ is not necessarily in $C^2([t,T]\times X\times U)$, in order to apply Dynkin's formula (18), we also need to invoke [4, Lemma 5.1] that for every $\delta > 0$ there exists $V^{(\varepsilon,\delta)} \in C^2([t,T]\times X\times U)$ such that

$$\|V^{(\varepsilon,\delta)} - V^{(\varepsilon)}\| < \delta \tag{50}$$

$$\|\mathscr{A} V^{(\varepsilon,\delta)}\| \leq \|\mathscr{A} V^{(\varepsilon)}\| + \delta \tag{51}$$

$$\mathscr{A} V^{(\varepsilon,\delta)} + l \geq -\delta \qquad \text{on } [t+\delta, T-\delta]\times X^0\times U \tag{52}$$

$$\|V_\partial^{(\varepsilon,\delta)}\| \leq \|V_\partial^{(\varepsilon)}\| + \delta \tag{53}$$

$$V^{(\varepsilon,\delta)}(s,x) - \ell(s,x) \leq \delta \quad (s,x) \in [t+\delta, T-\delta]\times X^\partial \tag{54}$$

$$\|V_T^{(\varepsilon,\delta)}\| \leq \|V_T^{(\varepsilon)}\| + \delta \tag{55}$$

$$V^{(\varepsilon,\delta)}(T,x) - L(x) \leq \delta \qquad x \in X \tag{56}$$

Then by (18),

$$
\begin{aligned}
V^{(\varepsilon,\delta)}(t,x) &= \big\langle V_{\partial}^{(\varepsilon,\delta)}, \eta_0 \big\rangle + \big\langle V_T^{(\varepsilon,\delta)}, \kappa_0 \big\rangle - \big\langle \mathscr{A} V^{(\varepsilon,\delta)}, \mu_0 \big\rangle \\
&\le \int_{[t+\delta, T-\delta)\times X^{\partial}} \ell \, d\eta_0 + \delta \int_{[t+\delta, T-\delta)\times X^{\partial}} d\eta_0 \\
&\quad + \|V_{\partial}^{(\varepsilon,\delta)}\| \int_{\{[t,t+\delta)\cup[T-\delta,T)\}\times X^{\partial}} d\eta_0 \\
&\quad + \int_X L \, d\kappa_0 + \big(\delta + \|V_T^{(\varepsilon,\delta)}\|\big) \int_X d\kappa_0 \\
&\quad + \int_{[t+\delta, T-\delta)\times X^0 \times U} l \, d\mu_0 + \delta \int_{[t+\delta, T-\delta)\times X^0 \times U} d\mu_0 \\
&\quad + \|\mathscr{A} V^{(\varepsilon,\delta)}\| \int_{\{[t,t+\delta)\cup[T-\delta,T)\}\times X^0\times U} d\mu_0 \qquad (57)
\end{aligned}
$$

and hence,

$$
\begin{aligned}
V^{(\varepsilon,\delta)}(t,x) &\le \big\langle l, \mu_0 \big\rangle + \big\langle \ell, \eta_0 \big\rangle + L, \kappa_0 \big\rangle \\
&\quad + \delta \cdot 2\big(1 + \|\mathscr{A} V^{(\varepsilon,\delta)}\| + \|V_{\partial}^{(\varepsilon,\delta)}\| + \|V_T^{(\varepsilon,\delta)}\|\big)(T-t+1)
\end{aligned} \qquad (58)
$$

Employing $\|V - V^{(\varepsilon,\delta)}\| < \varepsilon + \delta$ obtained from (46) and (50) and choosing first $\varepsilon$ then $\delta$ sufficiently small, we arrive at

$$
V(t,x) \le \big\langle l, \mu_0 \big\rangle + \big\langle \ell, \eta_0 \big\rangle + \big\langle L, \kappa_0 \big\rangle \qquad (59)
$$

that is in contradiction with the hypothesis (45). Therefore, the equivalence (44) holds true. ∎

The main result of this section is summarized in the following theorem

*Theorem 4.5:* For every $(t,x) \in [0,t] \times X$, the value function (3) can be computed from

$$
\begin{aligned}
V(t,x) = \sup \Big\{ & v(t,x) : v \in C^2([0,T]\times X), \\
& \mathscr{A} v + l \ge 0, \quad v^{\partial} - \ell \le 0, \quad v^T - L \le 0 \Big\}
\end{aligned} \qquad (60)
$$

In other words, the value function can be identified as the upper envelope of the smooth subsolutions of the Hamilton-Jacobi and the associated boundary value inequalities.

## V. Numerical Implementation

### A. Polynomial Representation

We recall from Stone–Weierstrass theorem (see e.g. [22]) that over the compact domain $[0,T]\times X \subset \mathbb{R}^{n+1}$, the algebra of all polynomials, $\mathbb{R}\big[s, x^{(1)}, \cdots, x^{(n)}\big]$, is dense in $C([0,T]\times X)$ and, consequently, in $C^2([0,T]\times X)$. Therefore, the family of test functions $v \in C^2([0,T]\times X)$ in (60) can be reduced to $v \in \mathbb{R}[s,x] \equiv \mathbb{R}\big[t, x^{(1)}, \cdots, x^{(n)}\big]$.

Moreover, if the functions $f, g, l, \ell, L$ are polynomials in their respective arguments, then the constraints $\mathscr{A} v + l \ge 0$, $v^{\partial} - \ell \le 0$, and $v^T - L \le 0$ can be replaced by algebraic relations on the coefficients of monomials in the representation of these polynomials. To illustrate this point, Consider the following example.

*Example 5.1:* Consider the stochastic system with scalar dynamics possessing a polynomial drift and a constant diffusion

$$
dx_s = f(x_s,u_s)\,dt + g\,dw_s = \Bigg(\sum_{j\ge 0, k\ge 0} f_{j,k} x_s^j u_s^k\Bigg) dt + g\,dw_s \qquad (61)
$$

over $X = [x_{\min}, x_{\max}]$ and $U = [u_{\min}, u_{\max}]$ and the optimal control problem for the cost (2) with $l(x_s,u_s) = \sum_{j\ge 0,k\ge 0} l_{j,k} x_s^j u_s^k$, $\ell(\theta, x_{\min}) = \ell(\theta, x_{\max}) = \sum_{i\ge 0} \ell_i \theta^i$, and $L(x_T) = \sum_{j\ge 0} L_j x_T^j$. Every test function $v \in \mathbb{R}[s,x]$ is represented as

$$
v(s,x_s) = \sum_{i\ge 0, j\ge 0} v_{i,j} s^i x_s^j \qquad (62)
$$

and the act of $\mathscr{A}$ on $v$ is given as

$$
\begin{aligned}
\mathscr{A} v(s,x_s) &= \frac{\partial v}{\partial s} + f(x_s,u_s)\frac{\partial v}{\partial x_s} + \frac{1}{2} g^2 \frac{\partial^2 v}{\partial x_s^2}. \\
&= \sum_{i\ge 1, j\ge 0} i \cdot v_{i,j} s^{i-1} x_s^j \\
&\quad + \Bigg(\sum_{j\ge 0,k\ge 0} f_{j,k} x_s^j u_s^k\Bigg)\Bigg(\sum_{i\ge 0, j\ge 1} j\cdot v_{i,j} s^i x_s^{j-1}\Bigg) \\
&\quad + \frac{1}{2} g^2 \cdot \Bigg(\sum_{i\ge 0, j\ge 2} (j-1) j \cdot v_{i,j} s^i x_s^{j-2}\Bigg) \\
&= \sum_{i\ge 0, j\ge 0} i\cdot v_{i+1,j} s^i x_s^j + \sum_{i\ge 0, j\ge 0, k\ge 0} p_{i,j,k} s^i x_s^j u_s^k \\
&\quad + \sum_{i\ge 0, j\ge 0} \frac{1}{2} g^2 \cdot (j+1)(j+2)\cdot v_{i,j+2} s^i x_s^j \qquad (63)
\end{aligned}
$$

where

$$
p_{i,j,k} := \sum_{\alpha+\beta=j} f_{\alpha,k} v_{i,\beta+1} \qquad (64)
$$

Notice that the above relation must hold regardless of the selection of $(s,x) \in [t,T]\times X^0$ and hence, we use the general notations $x, u$ instead of $x_s, u_s$. To avoid confusion with the specific point of interest $(t,x) \equiv (t,x_t)$, we keep using $s$ to refer to a general time variable. Invoking Theorem 4.5 for the system in this example, we deduce

$$
\begin{aligned}
V(t,x_t) = \sup \Bigg\{ & v(t,x_t) \equiv \sum_{i\ge 0, j\ge 0} v_{i,j} t^i x_t^j : \qquad (65) \\
& \sum_{j\ge 0,k\ge 0} l_{j,k} x^j + \sum_{i\ge 0,j\ge 0} i\cdot v_{i+1,j} s^i x^j + \sum_{i\ge 0,j\ge 0,k\ge 0} p_{i,j,k} s^i x^j u^k \\
& \quad + \sum_{i\ge 0,j\ge 0} \frac{1}{2} g^2 \cdot (j+1)(j+2)\cdot v_{i,j+2} s^i x^j \ge 0 \qquad (66) \\
& \sum_{i\ge 0,j\ge 0} v_{i,j} x_{\max}^j s^i - \sum_{i\ge 0} \ell_i \theta^i \le 0 \qquad (67) \\
& \sum_{i\ge 0,j\ge 0} v_{i,j} x_{\min}^j s^i - \sum_{i\ge 0} \ell_i \theta^i \le 0 \qquad (68) \\
& \sum_{i\ge 0,j\ge 0} v_{i,j} x^j T^i - \sum_{j\ge 0} L_j x^j \le 0 \qquad (69)
\end{aligned}
$$

Notice, first, that the above problem is an infinite dimensional optimization (over the space of all coefficients $v_{i,j}$) and, second, even with the restriction of polynomial degrees (e.g. $i+j \le d$), the number of coefficients grows exponentially in $d$.

*B. Putinar's Positivstellensatz [9, Chapter 2]*

Let $\Sigma[x] \subset \mathbb{R}[x]$ denote the space of real sum of squares polynomials in $x \in \mathbb{R}^n$. Let $\left(h_X^{(i)}(x)\right)_{i=1}^m \subset \mathbb{R}[x]$ be such that the basic semi-algebraic set

$$X := \{x \in \mathbb{R}^n : h_X^{(i)}(x) \geq 0, i = 1, \cdots, m\} \tag{70}$$

is compact. The quadratic module generated by the family $h_X = \{h_X^{(i)}(x)\}_{i=1}^m$ is defined by

$$Q(h_X) := \left\{ q^{(0)}(x) + \sum_{i=1}^m q^{(i)}(x) \cdot h_X^{(i)}(x) \; : \; \left(q^{(i)}(x)\right)_{i=0}^m \subset \Sigma[x] \right\} \tag{71}$$

If $w(x) \in \mathbb{R}[x]$ is strictly positive on $X$ and there exists $p(x) \in Q(h_X)$ such that the set $\{x \in \mathbb{R}^n : p(x) \geq 0\}$ is compact, then

$$w(x) = w^{(0)}(x) + \sum_{i=1}^m w^{(i)}(x) \cdot h_X^{(i)}(x) \tag{72}$$

for some sum of squares polynomials $w^{(i)} \in \Sigma[x]$.

*C. Sum of Squares Representations*

Let the state and input value sets $X$ and $U$ be compact basic semi-algebraic domains of the form

$$X = \left\{ x \in \mathbb{R}^n : h_X^{(j)}(x) \geq 0, 1 \leq j \leq n_X \right\}, \tag{73}$$

$$U = \left\{ u \in \mathbb{R}^m : h_U^{(k)}(u) \geq 0, 1 \leq k \leq n_U \right\}, \tag{74}$$

with $\left(h_X^{(j)}(x)\right)_{j=1}^{n_X} \in \mathbb{R}[x]$ and $\left(h_U^{(k)}(u)\right)_{k=1}^{n_U} \in \mathbb{R}[u]$. In order to satisfy Putinar's conditions, we assume that polynomials of the form $h_X^{(j)}(x) = R_X^2 - \|x\|_2^2$, and $h_U^{(k)}(u) = R_U^2 - \|u\|_2^2$ are either a member of $\left(h_X^{(j)}(x)\right)_{j=1}^{n_X}$ and $\left(h_U^{(k)}(u)\right)_{k=1}^{n_U}$ or are auxiliarily added with $R_X$ and $R_U$ large enough not to change $X$ and $U$. We remark that the boundary $X^\partial$ has the form

$$X^\partial = \bigcup_{i=1}^{n_X} \left\{ x \in \mathbb{R}^n : h_X^{(i)}(x) = 0, h_X^{(j)}(x) \geq 0, j \neq i \right\} \tag{75}$$

Moreover, we define $h_T(s) \in \mathbb{R}[s]$ such that

$$[t, T] = \{s \in \mathbb{R} : h_T(s) = (s - t) \cdot (T - s) \geq 0\} \tag{76}$$

Invoking Putinar's Positivstellensatz, we substitute the positivity conditions for $\mathscr{A} v + l$, $\ell - v^\partial$ and $L - v^T$ with their memberships, respectively, in $Q(h_T, h_X, h_U)$, $Q(h_T, h_X)$ and $Q(h_X)$.

*D. Lasserre's Hierarchy*

Let $\mathbb{R}_k[x]$ denote the vector space of real multivariate polynomials of total degree less than or equal to $k$. Let also $\Sigma_k[x] \subset \mathbb{R}_k[x]$ denote the space of sum of squares polynomials of at most degree $k$. Define

$$Q_{2k}(h_X) := \left\{ q_X^{(0)}(x) + \sum_{i=1}^{n_X} q_X^{(i)}(x) \cdot h_X^{(i)}(x) : \left(q_X^{(i)}\right)_{i=0}^{n_X} \subset \Sigma_{2k}[x] \;,\; \left(q_X^{(i)} h_X^{(i)}\right)_{i=1}^{n_X} \in \mathbb{R}_{2k}[x] \right\} \tag{77}$$

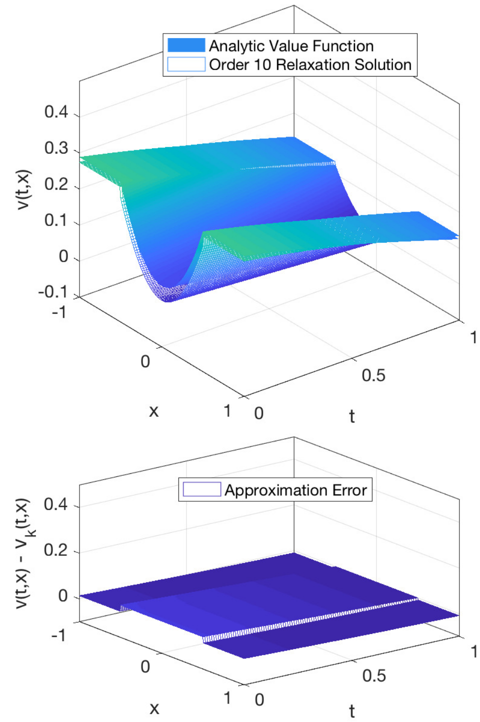


Fig. 1: Top: The comparison of the analytic value function $V(t,x)$ and its relaxation degree $2k = 10$ subsolution $v_{10}(t,x)$. Bottom: The approximation error for the degree 10 relaxation.

Then the degree $2k$ relaxation function is defined to be

$$V_{(2k)} := \max \Big\{ v(t, x_t) : \; \mathscr{A} v + l \in Q_{2k}(h_T, h_X, h_U) \\ \ell - v^\partial \in Q_{2k}(h_T, h_X) \\ L - v^T \in Q_{2k}(h_X) \Big\} \tag{78}$$

where the maximum is taken over all SOS coefficients $\left(q_X^{(i)}\right)_{i=0}^{n_X}$, $\left(q_U^{(i)}\right)_{i=0}^{n_U}$ and $\left(q_T^{(i)}\right)_{i=0}^{1}$.

*Theorem 5.2:* The sequence $\{V_{(2k)}\}_{k=\min\deg(f,g,l,\ell,L)}^\infty$ is monotonically increasing, i.e., $V_{2(k+1)} \geq V_{2k}$, and

$$V(t,x) = \lim_{k \to \infty} V_{2k} \tag{79}$$

*Example 5.3:* Consider the scalar system

$$dx = (x + u)dt + \frac{1}{5} dw \tag{80}$$

with $X = [-0.5, 0.5]$, and $U = [-1, 1]$; and the associated optimal control problem with

$$J(t, x, \boldsymbol{u}) = \mathbb{E}_{t,x}^{\boldsymbol{u}} \Big\{ \int_t^{\min\{\theta, T\}} \frac{1}{2} u_s^2 \, ds + \\ + \mathbb{I}_{[t,T)}(\theta) \cdot \ell(\theta) + \mathbb{I}_{[t,T)}^c(\theta) \cdot \frac{1}{2} x_T^2 \Big\}, \tag{81}$$

We know that the value function for the unbounded (not killed) Markov process is given by

$$V(t,x) = \frac{x^2}{e^{-2(1-t)} + 1} + \frac{1}{50} \left( \ln\left( \frac{1 + e^{2(1-t)}}{2} \right) + 2(1-t) \right) \tag{82}$$

| | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ | $x^9$ | $x^{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t^0$ | 0.0687 | 0 | 0.8808 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $t^1$ | -0.0752 | 0 | -0.2100 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| $t^2$ | 0.0042 | 0 | -0.1599 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| $t^3$ | 0.0021 | 0 | -0.0518 | 0 | 0 | 0 | 0 | 0 | | | |
| $t^4$ | 0.0005 | 0 | 0.0139 | 0 | 0 | 0 | 0 | | | | |
| $t^5$ | -0.0001 | 0 | 0.0232 | 0 | 0 | 0 | | | | | |
| $t^6$ | -0.0002 | 0 | 0.0095 | 0 | 0 | | | | | | |
| $t^7$ | -0.0001 | 0 | -0.0012 | 0 | | | | | | | |
| $t^8$ | 0.0000 | 0 | -0.0034 | | | | | | | | |
| $t^9$ | 0.0000 | 0 | | | | | | | | | |
| $t^{10}$ | 0.0000 | | | | | | | | | | |

TABLE I: Taylor expansion of the value function

| | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ | $x^9$ | $x^{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t^0$ | 0.0287 | 0.0001 | 0.8803 | 0.0029 | -0.0084 | 0.0261 | -0.0502 | -0.0022 | 0.0167 | 0.0626 | 0.0589 |
| $t^1$ | -0.0352 | -0.0005 | -0.2069 | -0.0113 | 0.0408 | -0.0383 | 0.1184 | 0.0615 | 0.0793 | 0.0774 | |
| $t^2$ | 0.0040 | 0.0017 | -0.1702 | 0.0154 | -0.0670 | -0.0025 | -0.0098 | -0.0032 | 0.0293 | | |
| $t^3$ | 0.0027 | -0.0038 | -0.0332 | -0.0078 | 0.0359 | 0.0129 | -0.0855 | -0.1052 | | | |
| $t^4$ | -0.0010 | 0.0052 | -0.0044 | -0.0039 | -0.0003 | -0.0283 | -0.0245 | | | | |
| $t^5$ | 0.0025 | -0.0032 | 0.0232 | 0.0079 | -0.0159 | 0.0342 | | | | | |
| $t^6$ | -0.0023 | -0.0015 | 0.0360 | 0.0004 | 0.0194 | | | | | | |
| $t^7$ | -0.0005 | 0.0040 | -0.0307 | -0.0037 | | | | | | | |
| $t^8$ | 0.0028 | -0.0026 | 0.0057 | | | | | | | | |
| $t^9$ | -0.0022 | 0.0006 | | | | | | | | | |
| $t^{10}$ | 0.0006 | | | | | | | | | | |

TABLE II: Coefficients of the degree 10 relaxation of the value function

To be able to make a direct comparison with an analytic solution, we would like to set the termination (killing) cost equal to $V(t, x^{\partial})$. Since this cost is not polynomial in $t$, we define $\ell$ to be its 5th order Taylor expansion, i.e.

$$\begin{aligned}\ell(\theta) = 0.0061916470059\theta^5 + 0.0040208691132\theta^4 + \\ - 0.01283351403\theta^3 - 0.040911889912\theta^2 + \\ - 0.0973217753\theta + 0.274900044 \quad (83)\end{aligned}$$

The true value function and its polynomial approximation with degree 12 relaxations are illustrated in Figure 1. The coefficients of the Taylor series expansion of the value function and the coefficients of its degree 10 relaxed approximate representation are displayed in Table I and II. As seen from Figure 1, the smallest relaxation degree provides an approximation of the value function with relatively small error. However, it is worth noting that non-zero coefficients contribute significantly to errors. While a part of the error is caused by the degree 5 Taylor expansion of the leaving cost $\ell$, a part of the error is also contributed directly from the numerical solver.

## VI. Concluding Remarks

The class of killed Markov processes considered in this paper covers a variety of stochastic control problems in control theory. For instance, with an appropriate definition of pushforward the leaving measure and extension of the dynamics upon arrival on the boundary manifold, one can extend . the framework to stochastic hybrid systems. One of the significant aspects of such a generalisation, in comparison with variational methods for stochastic hybrid systems [23], is that the optimality conditions can be established without further assumptions on the relationship between the diffusion direction and the boundary surface of the evolution space.

One the numerical side, however, the results shall be employed more cautiously. First, as observed in Example 5.3, there exist unavoidable numerical errors since e.g. sparsity structures are not generally known to be fed into algorithms. Moreover, convergence becomes slow as the degree of relaxation $2k$ becomes large. Last but not least, errors resulting from large relative magnitude of second order derivatives shall be taken into account, particularly for systems with large diffusion coeffients.

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