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# A Mean Field Game - Hybrid Systems Approach to Optimal Execution Problems in Finance with Stopping Times

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*Abstract*— **The paper combines two major contemporary systems and control methodologies to obtain a unique $\epsilon$-Nash equilibrium for optimal execution problems within the stock market, namely Mean Field Game (MFG) theory and Hybrid Optimal Control (HOC) theory. Following standard financial models, the stock market is studied in this paper as a large population non-cooperative game where each trader has stochastic linear dynamics with quadratic costs. We consider the case where there exists one major trader with significant influence on market movements together with a large number of minor traders (within two subpopulations), each with individually asymptotically negligible effect on the market. The traders are coupled in their dynamics and cost functions by the market's average trading rate (a component of the system mean field) and the hybrid feature enters via the indexing of the cessation of trading by one or both subpopulations of minor traders by discrete states. Optimal stopping time strategies together with best response trading policies for all traders are established with respect to their individual cost criteria by an application of LQG HOC theory.**

## I. INTRODUCTION

Mean Field Game (MFG) systems theory establishes the existence of approximate Nash equilibria together with the corresponding individual strategies for stochastic dynamical systems in games involving a large number of agents. The equilibria are termed $\epsilon$-Nash equilibria and are generated by the local, limited information feedback control actions of each agent in the population, where the feedback control actions constitute the best response of each agent with respect to the precomputed behaviour of the mass of agents and where the approximation error converges to zero as the population size goes to infinity.

The analysis of this set of problems originated in [1]–[3] (see [4]), and independently in [5], [6]. In [7] and [8] the authors analyse and solve the linear quadratic systems case where there is a major agent (i.e. non-asymptotically vanishing as the population size goes to infinity) together with a population of minor agents (i.e. individually asymptotically negligible). The existence of $\epsilon$-Nash equilibria is established together with the individual agents' control laws that yield the equilibria [8]. The partially observed MFG theory for nonlinear and linear quadratic systems with major and minor agents has been developed in [9]–[15].

The notion of Hybrid Systems (HS) is used to describe control systems for which (i) the state has continuous and discrete components and (ii) the continuous component of the state evolves in "continuous time" and the discrete component evolves in "discrete time". While the optimal control of deterministic hybrid systems has been extensively studied in the literature (see e.g. [16]–[27]), the optimal control of stochastic hybrid systems, i.e. control systems that involve the interaction of continuous dynamics, discrete dynamics and stochastic diffusions, has been the subject of a limited number of studies [28]–[31]. In [30], in particular, first order variational analysis is performed on the stochastic hybrid optimal control problem via the needle variation methodology and the necessary optimality conditions are established in the form of the Stochastic Hybrid Minimum Principle (SHMP). The results are specialized in [31] to a class of Linear Quadratic Gaussian Hybrid Optimal Control Problems (LQG-HOCP) for which the Hamiltonian boundary conditions are path-independent and therefore, the corresponding stochastic Riccati equations are independent from the realization of stochastic diffusion terms.

Optimal execution problems have been addressed in the literature (see e.g. [32]–[36]) in which an agent must liquidate or acquire a certain amount of shares over a prespecified time horizon while it balances the price impact and the price uncertainty, and maximizes its final wealth. This problem with the linear models in [32] was formulated as for the nonlinear major minor (MM) MFG model in [37], and partially observed MM LQG MFG theory was first applied in [38], [39] for the case in which traders have partial observations of the market states. In this paper, the stock market consists of an institutional investor, interpreted as the major agent, aims to liquidate a specific amount of shares, and a large population high frequency traders (HFTs), interpreted as minor agents, who wish to liquidate or acquire a certain amount of shares within a specific time horizon. The traders are coupled in their dynamics and cost functions by the market's average trading rate (a component of the system mean field) and the hybrid feature enters via the indexing of the cessation of trading by one or both subpopulations of minor traders by discrete states. This work combines two major contemporary systems and control techniques: MFG theory and Hybrid Optimal Control (HOC) theory to establish optimal stopping time strategies together with best response trading policies for all agents with respect to their individual cost criteria which yield a unique $\epsilon$-Nash equilibria for the market.

We note major trader (respectively, minor trader), and institutional trader (respectively, HFT) are used interchangeably in this paper.

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## II. Trading Dynamics of Agents in the Market

As stated in the Introduction, the institutional investor is considered as a major agent in the mean field model of the market which liquidates its shares and the HFTs are considered as minor agents, where two types of them are considered: acquirers $\mathcal{A}_a$ with the population of $N_a$ and liquidators $\mathcal{A}_l$ with the population of $N_l$, such that $N_a + N_l = N$. All agents trade over the interval $[0, T]$, and minor agents are allowed to stop trading at an optimal time $t_s^i \leq T$. It will be shown in Section III that the optimal stopping time policy for each agent is $\mathcal{F}_t$-independent, and depends only on its dynamical parameters. In this paper, for simplicity of exposition the dynamical parameters for all minor traders in their respective type are the same, and hence the stopping times are the same for all agents of each population. Employing the trading model in [32], the trading dynamics of the major agent and any generic minor agent in the market are described by the linear time evolution of the (i) inventories, (ii) trading rates and (iii) prices while the bilinear cash process appears in the quadratic performance function for each agent.

### A. Inventory Dynamics

It is assumed that the institutional investor liquidates its inventory of shares, $Q_0(t)$, by trading at a rate $\nu_0(t)$ during the trading period $[0, T]$. Hence the major agent's inventory dynamics is given by

$$dQ_0(t) = \nu_0(t)dt + \sigma_0^Q dw_0^Q, \quad 0 \leq t \leq T,$$

where $w_0^Q$ is a Wiener process modeling the noise in the inventory information that the institutional trader collects from its branches in different locations; $\sigma_0^Q$ is a positive scalar and we assume that $Q_0(0) \gg 1$. The same dynamical model is adopted for the trading dynamics of a generic HFT

$$dQ_i(t) = \nu_i(t)dt + \sigma_i^Q dw_i^Q,$$

where for a minor acquirer trader $\mathcal{A}_i \in \mathcal{A}_a$, $0 \leq t \leq t_s^a$, and correspondingly for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, $0 \leq t \leq t_s^l$. The Wiener process $w_i^Q$ models the HFT's information noise, $\sigma_i^Q$ is a positive scalar, $\nu_i(t)$ is the agent's rate of trading which can be positive or negative depending on whether the agent is acquirer or liquidator, respectively; $Q_i(t)$ is the minor liquidator's remaining shares at time $t$, or the shares the minor acquirer has bought until time $t$. However, the initial share stock of the HFTs, $\{Q_i(0), 1 \leq i \leq N_a + N_l\}$, are not considered to be large, furthermore they are not motivated to retain shares and are assumed to trade them quickly.

We assume that the trading rate of the major agent is controlled via $u_0(t)$ as

$$d\nu_0(t) = u_0(t)dt, \quad 0 \leq t \leq T,$$

where the trading strategy $u_0(t)$ can be seen to be the trading acceleration of the major trader. Correspondingly, $u_i(t)$ controls the trading rate of minor agent, $\mathcal{A}_i$, by

$$d\nu_i(t) = u_i(t)dt,$$

where again for a minor acquirer trader $\mathcal{A}_i \in \mathcal{A}_a$, $0 \leq t \leq t_s^a$, and correspondingly for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, $0 \leq t \leq t_s^l$, and $u_i(t)$ is the trading acceleration of the minor acquirer or liquidator.

### B. Price Dynamics

The trading rate of the major agent and the average trading rate of the minor agents give rise to the fundamental asset price which models the permanent effect of agents' trading rates on the market price. Further, each agent has a temporary effect on the asset price which only persists during the action of the trade and which determines the execution price, that is to say the price at which each agent can trade.

*1) Fundamental Asset Price:* We model the dynamics of the fundamental asset price, as seen from the major agent's viewpoint, by

$$dF_0(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + \sigma dw_0^F(t), \ 0 \leq t \leq T,$$

where $N_t$ is the number of minor agents trading at time $t$, $\nu^{N_t}(t) = \frac{1}{N_t}\sum_{i=1}^{N_t}\nu_i(t)$ is the average trading rate of the minor agents trading at time $t$. The Wiener process $w_0^F(t)$ models the aggregate effect of all traders in the market which - unlike the major and minor agents $\mathcal{A}_0$, $\mathcal{A}_i$, - have no complete or partial observations on any of the state variables appearing in the dynamical market model (these are termed uninformed traders). Further, $\sigma$ denotes the intensity of the market volatility and $\lambda_0, \lambda \geq 0$ denote the strength of the linear permanent impact of the major and minor agents' trading on the fundamental asset price, respectively. Similarly, we model the fundamental asset price dynamics, as seen by a minor agent $\mathcal{A}_i$, by

$$dF_i(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + \sigma dw_i^F(t),$$

where $0 \leq t \leq t_s^a$, for $\mathcal{A}_i \in \mathcal{A}_a$, and $0 \leq t \leq t_s^l$, for $\mathcal{A}_i \in \mathcal{A}_l$, $\nu^{N_t}(t) = \frac{1}{N_t}\sum_{i=1}^{N_t}\nu_i(t)$ is again the average trading rate of the $N_t$ minor agents trading at $t$, and the Wiener process, $w_i^F(t)$, represents the mass effect of all uninformed traders in the market. The time differences between agents in getting data from fast changing limit order book make the Wiener processes, $w_i^F$, $0 \leq i \leq N_a + N_l$ independent.

*2) Execution Price:* The major agent's execution price $S_0(t)$ evolution is assumed to be given by

$$dS_0(t) = dF_0(t) + a_0 d\nu_0(t), \quad 0 \leq t \leq T, \tag{1}$$

where $a_0 \geq 0$ is the temporary impact strength of the major agent on fundamental asset price. Likewise, a minor agent's execution price, $S_i(t)$, is assumed to evolve by

$$dS_i(t) = dF_i(t) + a d\nu_i(t), \tag{2}$$

where $0 \leq t \leq t_s^a$, for $\mathcal{A}_i \in \mathcal{A}_a$, and $0 \leq t \leq t_s^l$, for $\mathcal{A}_i \in \mathcal{A}_l$, and $a$ models the temporary impact of a minor agent's trading on its execution price.

### C. Cash Process

The cash processes for the major agent and a generic minor agent, $Z_0(t)$, $Z_i(t)$, respectively, are given by

$$dZ_0(t) = -S_0(t)dQ_0(t), \quad 0 \leq t \leq T, \tag{3}$$

$$dZ_i(t) = -S_i(t)dQ_i(t), \quad \begin{matrix} \text{for } \mathcal{A}_i \in \mathcal{A}_a, \ 0 \leq t \leq t_s^a \\ \text{for } \mathcal{A}_i \in \mathcal{A}_l, \ 0 \leq t \leq t_s^l \end{matrix}, \tag{4}$$

where $Z_0(t)$, and $Z_i(t)$ for $\mathcal{A}_i \in \mathcal{A}_l$ are the cash obtained through liquidation of shares, and $Z_i(t)$, for $\mathcal{A}_i \in \mathcal{A}_a$ is the cash paid for acquisition of shares up to time $t$. We note that the value of $dQ_0(t)$ in a stock sale (respectively, buy) is negative (respectively, positive) and hence for positive $S_0(t)$, $Z_0(t)$ increases (respectively, decreases).

### D. Performance Function

*1) Major Liquidator:* The objective for the major trader is to liquidate $\mathcal{N}_0$ shares and maximize the cash it holds at the end of the trading horizon, i.e. maximize $Z_0(T)$, and if the remaining inventory at the final time $T$ is $Q_0(T)$, it can liquidate it at a lower price than the market asset price reflected at cost function by $Q_0(T)(F_0(T)-\alpha Q_0(T))$. Further, the major trader's utility in minimizing the inventory over the period $[0,T]$ is modeled by including the penalty $\phi\int_0^T Q_0^2(s)ds$ in its objective function, and the utility of avoiding very high execution prices, large trading intensities and large trading accelerations by including the terms $\epsilon S_0^2(T)$, $\int_0^T \delta S_0^2(s)ds$, $\beta\nu_0^2(T)$, $\int_0^T \theta\nu_0^2(s)ds$ and $\int_0^T R_0u_0^2(s)ds$ in the objective function. Therefore, its cost function to be minimized is given by

$$J_0(u_0,u_{-0}) = \mathbb{E}\Big[-rZ_0(T)-pQ_0(T)\big(F_0(T)-\alpha Q_0(T)\big) + \epsilon S_0^2(T) + \beta\nu_0^2(T) + \int_0^T \big(\phi Q_0^2(s) + \delta S_0^2(s) + \theta\nu_0^2(s) + R_0u_0^2(s)\big)ds\Big], \quad (5)$$

where $r$, $p$, $\alpha$, $\epsilon$, $\beta$, $\phi$, $\delta$, $\theta$, and $R_0$ are positive scalars, and $u_{-0} := (u_1,u_2,...,u_N)$ are trading strategies of the minor traders. Note that for larger values of $\phi$ the trader attempts to liquidate its inventory more quickly.

*2) Minor Liquidator :* In a similar way, the objective function to be minimized for a liquidator HFT who wants to liquidate $\mathcal{N}_l$ shares over the interval $[0,T]$ with the stopping time $0 \le t_s^l \le T$ is given by

$$J_i(u_i,u_{-i}) = \mathbb{E}\Big[-r_lZ_i(t_s^l)-p_lQ_i(t_s^l)\big(F_i(t_s^l)-\psi_lQ_i(t_s^l)\big) + \xi_lS_i^2(t_s^l) + \mu_l\nu_i^2(t_s^l) + \int_0^{t_s^l} \big(\kappa_lQ_i^2(s) + \gamma_lS_i^2(s) + \varrho_l\nu_i^2(s) + R_lu_i^2(s)\big)ds\Big], \text{ for } \mathcal{A}_i \in \mathcal{A}_l \quad (6)$$

where $r_l$, $p_l$, $\psi_l$, $\xi_l$, $\mu_l$, $\kappa_l$, $\gamma_l$, $\varrho_l$ and $R_l$ are positive scalars, and $u_{-i} := (u_0,u_1,...,u_{i-1},u_{i+1},...,u_N)$. Note that $\mathcal{N}_l \ll \mathcal{N}_0$.

*3) Minor Acquirer:* The objective for a minor acquirer is to buy $\mathcal{N}$ shares during the trading horizon $[0,T]$. Given that it stops trading at $t_s^a \le T$, it also wishes to minimize the execution cost including the cash $Z_i(t_s^a)$ paid up to time $t_s^a$, and the cash must be paid at time $t_s^a$ to buy the remaining shares at once at a higher price than the market's asset price, i.e. $\big(\mathcal{N}-Q_i(t_s^a)\big)\big(F_i(t_s^a)+\psi_a(\mathcal{N}-Q_i(t_s^a)\big)$. It is also intended to avoid high execution prices, large trading intensities and large trading accelerations modeled by including $\xi_aS_i^2(t_s^a)+\mu_a\nu_i^2(t_s^a)+\int_0^{t_s^a}\big(\gamma_aS_i^2(s)+\varrho_a\nu_i^2(s)+R_Au_i^2(s)\big)ds$ in its objective function

$$J_i(u_i,u_{-i}) = \mathbb{E}\Big[p_a(\mathcal{N}-Q_i(t_s^a))\big(F_i(t_s^a)+\psi_a(\mathcal{N}-Q_i(t_s^a))\big) + r_aZ_i(t_s^a) + \xi_aS_i^2(t_s^a) + \mu_a\nu_i^2(t_s^a) + \int_0^{t_s^a} \big(\kappa_a(\mathcal{N}-Q_i(s))^2 + \gamma_aS_i^2(s) + \varrho_a\nu_i^2(s) + R_au_i^2(s)\big)ds\Big], \ \mathcal{A}_i \in \mathcal{A}_a, \quad (7)$$

where $\int_0^{t_s^a}\kappa_a(\mathcal{N}-Q_i(s))^2ds$ is to penalize the agent for the remaining shares to be bought up to $t_s^a$ and to expedite the acquisition. The parameters $p_a$, $\psi_a$, $r_a$, $\xi_a$, $\mu_a$, $\kappa_a$, $\gamma_a$, $\varrho_a$, and $R_a$ are positive scalars and $u_{-i} := (u_0,u_1,...,u_{i-1},u_{i+1},...,u_N)$.

## III. HS-MFG Formulation of the Optimal Execution Problem

In this section we formulate the optimal execution problem in the MM LQG MFG framework.

### A. Discrete state association

In order to present the trading dynamics of the stock market in the stochastic hybrid systems framework of [30], [31], the discrete states $q_j,\ j=0,1,2$ are introduced, which correspond to the evolution of the market in the intervals $[t_j,t_{j+1})$, where $t_0=0$ is the initial time, $t_1$ and $t_2$ denote the stopping times of the first population and the second population respectively, and $t_3=T$ is the terminal time.

We remark that the HS-MFG problems studied in this paper lie within the class of hybrid LQG problems in [31] for which optimal switching strategies are $\mathcal{F}_t$-independent, and therefore, optimal stopping strategies depend only on the dynamical parameters of each population.

We associate the discrete state $q_0$ to the initial case where both the liquidator and acquirer populations are trading together with the major agent over the interval $[0,t_1)$.

The discrete state $q_1$ corresponds to the interval $[t_1,t_2)$ for which two situations can be considered: (i) the liquidator population stops at $t_1$ while the acquirer population is still trading, in which case $q_1 = q_a$, and (ii) the acquirer population stops at $t_1$ while the liquidator population is trading, which corresponds to $q_1=q_l$.

The discrete state $q_2$ represents the system over the interval $[t_2,T]$ after the second population of HFTs stops at $t_2$, and hence the major agent is trading in the absence of both populations.

The above discrete state association is summerized in the following table.

| Discrete State | | $\mathcal{A}_0$ | $\mathcal{A}_a$ | $\mathcal{A}_l$ |
|---|---|---|---|---|
| $q_0$ | | ✓ | ✓ | ✓ |
| $q_1$ | $q_a$ | ✓ | ✓ | × |
| | $q_l$ | ✓ | × | ✓ |
| $q_2$ | | ✓ | × | × |

## B. Finite Populations

*1) Major Agent:* The dynamics of the major trader in the market can be modeled as

$$\begin{aligned}
&d\nu_0(t) = u_0(t)dt,\\
&dQ_0(t) = \nu_0(t)dt + \sigma_0^Q dw_0^Q,\\
&dS_0(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + a_0u_0(t)dt + \sigma dw_0^F(t).
\end{aligned}$$

Let the major agent's state be denoted by $x_0 = [\nu_0, Q_0, S_0]^T$, then its dynamics can be expressed as

$$dx_0 = A_0x_0dt + B_0u_0dt + E_0x^{N_t}dt + D_0dw_0 \quad (8)$$

with the matrices

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 1 \\ 0 \\ a_0 \end{bmatrix},\ w_0 = \begin{bmatrix} w_0^Q \\ w_0^F \end{bmatrix}$$

$$E_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix},\ D_0 = \begin{bmatrix} 0 & 0 \\ \sigma_0^Q & 0 \\ 0 & \sigma \end{bmatrix}.$$

Note that in (8), $N_t$ takes the following values.

$$N_t = \begin{cases} N_a + N_l & \text{for } q_0, \\ N_a & \text{for } q_a, \\ N_l & \text{for } q_l, \\ 0 & \text{for } q_2. \end{cases} \quad (9)$$

The major trader's cost function (5) can also be described in terms of its states with replacing the final cash process by $\mathbb{E}[Z_0(T)] = -\mathbb{E}[\int_0^T S_0(s)\nu_0(s)ds]$, and the fundamental asset price $F_0(T)$ using (1). The equation (8) together with the cost function (5) form the stochastic LQG problem for the major trader. Note that the major trader is involved with the market's average trading rate in its dynamics while involved with the market's average selling rate in its cost function.

*2) Minor Liquidator:* Similarly, the stochastic optimal control problem for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, is given by the set of dynamical equations

$$\begin{aligned}
&d\nu_i(t) = u_i(t)dt,\\
&dQ_i(t) = \nu_i(t)dt + \sigma_i^Q dw_i^Q,\\
&dS_i(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + au_i(t)dt + \sigma dw_i^F.
\end{aligned}$$

Similar to the major trader, we define a generic minor trader's state vector as $x_i = [\nu_i, Q_i, S_i]^T$, and its dynamics can be written as

$$dx_i = A_lx_idt + B_lu_idt + E_lx^{N_t}dt + D_ldw_{l_i} \quad (10)$$

with

$$A_l = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad E_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix},\quad B_l = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}$$

$$G_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix},\ D_l = \begin{bmatrix} 0 & 0 \\ \sigma_i^Q & 0 \\ 0 & \sigma \end{bmatrix},\ w_{li} = \begin{bmatrix} w_i^Q \\ w_i^F \end{bmatrix}.$$

The quadratic cost function (6) can also be expressed in terms of the minor agent's state when the final cash process in (6) is replaced by $\mathbb{E}[Z_i(t_s^l)] = -\mathbb{E}[\int_0^{t_s^l} S_i(s)\nu_i(s)ds]$ using (4), and the fundamental asset price $F_i(t_s^l)$ is replaced using (2). The equations (10) and (6) form the stochastic LQG problem for a generic minor liquidator. Additionally, they show that a minor liquidator is coupled with the major agent's trading rate and the market's average trading rate in its dynamics while coupled with the market's average selling rate in its cost function.

*3) Minor Acquirer Agent:* The stochastic optimal control problem for a minor acquirer $\mathcal{A}_i \in \mathcal{A}_a$, is given by the set of dynamical equations

$$\begin{aligned}
&d\nu_i(t) = u_i(t)dt,\\
&dY_i(t) = -\nu_i(t)dt + \sigma_i^Q dw_i^Q,\\
&dS_i(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + au_i(t)dt + \sigma dw_i^F,
\end{aligned}$$

where $Y_i(t) = \mathcal{N}_a - Q_i(t)$ is the remaining shares at $t$ to be acquired until the end of trading horizon. We define a generic minor acquirer's state vector as $x_i = [\nu_i, Y_i, S_i]$, hence its dynamics in compact form would be

$$dx_i = A_ax_idt + B_au_idt + E_ax^{N_t}dt + D_adw_{a_i}, \quad (11)$$

where

$$A_a = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ E_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix},\ B_a = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}$$

$$G_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix},\ D_a = \begin{bmatrix} 0 & 0 \\ \sigma_i^Q & 0 \\ 0 & \sigma \end{bmatrix},\ w_{ai} = \begin{bmatrix} w_i^Q \\ w_i^F \end{bmatrix}.$$

Note that $N_t$ in (11) again takes values as in (9) over the trading horizon. Accordingly, the cost function for acquisition is given by

$$\begin{aligned}
J_i(u_i, u_{-i}) = \mathbb{E}\Big[&p_aY_i(t_s^a)\big(S_i(t_s^a) - a\nu_i(t_s^a) + \psi_aY_i(t_s^a)\big)\\
&+ \xi_aS_i^2(t_s^a) + \mu_a\big(\nu_i(t_s^a) - \rho_a\bar{\nu}(t_s^a)\big)^2 +\\
&\int_0^{t_s^a} \big(\kappa_aY_i(s)^2 + \gamma_aS_i^2(s) + \varrho_a\nu_i^2(s)\\
&- r_aS_i(s)\nu_i(s) + R_au_i^2(s)\big)ds\Big],\ \text{for } \mathcal{A}_i \in \mathcal{A}_a. \quad (12)
\end{aligned}$$

The set of equations (11)-(12) constitute the standard stochastic LQG problem for a minor acquirer. It can be seen that a generic minor acquirer interacts with the major agent's trading rate as well as the market's average trading rate through it dynamics, and with the market's average buying rate through its cost function.

## C. Mean Field Evolution

Following the LQG MFG methodology [7], the mean field, $\bar{x}$, is defined as the $L^2$ limit, when it exists, of the average of minor agents' states when population size goes to infinity

$$\bar{x}(t) = \lim_{N_t\to\infty} x^{N_t}(t) = \lim_{N\to\infty} \frac{1}{N_t}\sum_{i=1}^{N_t} x_i(t),\ a.s.$$

Now, if the control strategy for each minor agent is considered to have the general feedback form

$$u_i = L_1 x_i + L_2 x_0 + \sum_{j\neq i, j=1}^{N_t} L_4 x_j + L_3, \quad 1 \leq i \leq N_t, \quad (13)$$

then the mean field dynamics can be obtained by substituting (13) in the minor liquidator (respectively, acquirer) agents' dynamics (10) (respectively, (11)), and taking the average and then its $L^2$ limit as $N \to \infty$.

The set of mean field equations for the optimal execution problem can be written as

$$d\bar{x} = \bar{A}\bar{x}dt + \bar{G}x_0 dt + \bar{m}dt. \quad (14)$$

For $q_0$, $\bar{x} = [\bar{x}_a^T, \bar{x}_l^T]^T$ consists of the mean field $\bar{x}_l$ of the liquidator population, and the mean field $\bar{x}_a$ of the acquirer population. The matrices in (14) are defined as

$$\bar{A} = \begin{bmatrix} \bar{A}_a & \bar{A}_{al} \\ \bar{A}_{la} & \bar{A}_l \end{bmatrix}, \bar{G} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_l \end{bmatrix}, \bar{m} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_l \end{bmatrix}, \quad (15)$$

which shall be determined from consistency equations discussed in section III-E.

For $q_a$, $\bar{x} = \bar{x}_a$, and the matrices in (14) are given as

$$\bar{A} = \bar{A}_a, \quad \bar{G} = \bar{G}_a, \quad \bar{m} = \bar{m}_a. \quad (16)$$

For $q_l$, $\bar{x} = \bar{x}_l$, and the matrices in (14) are given by

$$\bar{A} = \bar{A}_l, \quad \bar{G} = \bar{G}_l, \quad \bar{m} = \bar{m}_l. \quad (17)$$

Finally, for $q_2$, $\bar{x} = 0$.

### D. Infinite Populations

Following the mean field game methodology with a major agent [7], [40] the hybrid optimal execution problem is first solved in the infinite population case where the average term in the finite population dynamics and cost function of each agent is replaced by its infinite population limit, i.e. the mean field. Then specializing to linear systems [7], the major agent's state is extended with the mean field, while the minor agent's state is extended with the mean field and the major agent's state; this yields LQG problems for each trader linked only through the mean field and the major agent's state. Then the main results of [7], [40] are (i) the existence of infinite population best response strategies which yield the Nash equilibria, and (ii) the infinite population best response strategies applied to the finite population system yield an $\epsilon$-Nash equilibria (see Theorem 3.1).

*1) Major Liquidator Agent:* The extended dynamics of the major agent in the infinite population, i.e. the dynamic for the $x_0^{ex,q_i}$ is given by

$$dx_0^{ex,q_j} = (\mathbb{A}_0^{q_j} x_0^{ex,q_j} + \mathbb{M}_0^{q_j} + \mathbb{B}_0^{q_j} u_0^{q_j})dt + \mathbb{D}_0^{q_j} dW_0, \quad (18)$$

$0 \leq j \leq 2$, and the cost function for the extended major agent's system would be

$$J_0(u_0, u_{-0}) = \mathbb{E}\Big[\|x_0^{ex,q_2}(T)\|^2_{\bar{\mathbb{P}}_0^{q_2}} + \sum_{j=0}^{2} \int_{t_j}^{t_{j+1}} \big(\|x_0^{ex,q_j}(s)\|^2_{\mathbb{P}_0^{q_j}} + \|u_0^{q_j}(s)\|^2_{R_0^{q_j}}\big) ds\Big], \quad (19)$$

where $t_0 = 0, t_3 = T$. Let matrix coefficients $P_0$, $\bar{P}_0$, respectively, associated with the running and final costs in (5) be given by

$$\bar{P}_0 = \begin{bmatrix} \beta & \frac{1}{2}pa_0 & 0 \\ \frac{1}{2}pa_0 & p\alpha & -\frac{1}{2}p \\ 0 & -\frac{1}{2}p & \epsilon \end{bmatrix}, P_0 = \begin{bmatrix} \theta & 0 & \frac{1}{2}r \\ 0 & \phi & 0 \\ \frac{1}{2}r & 0 & \delta \end{bmatrix},$$

then over the interval $[t_0, t_1)$, and in the discrete state $q_0$, the dynamics of the continuous state $x_0^{ex,q_0} = [x_0^T, \bar{x}_a^T, \bar{x}_l^T]^T$ is determined from (18) with

$$\mathbb{A}_0^{q_0} = \begin{bmatrix} A_0 & [E_0, E_0] \\ \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{M}_0^{q_0} = \begin{bmatrix} 0_{3\times1} \\ \bar{m} \end{bmatrix},$$

$$\mathbb{B}_0^{q_0} = \begin{bmatrix} B_0 \\ 0_{6\times1} \end{bmatrix}, \quad \mathbb{D}_0^{q_0} = \begin{bmatrix} D_0 & 0_{3\times6} \\ 0_{6\times3} & 0_{6\times6} \end{bmatrix}.$$

and $\mathbb{P}_0^{q_0}$ in (19) is given by

$$\mathbb{P}_0^{q_0} = [I_{3\times3}, 0_{3\times3}, 0_{3\times3}]^T P_0 [I_{3\times3}, 0_{3\times3}, 0_{3\times3}].$$

In case (i) where $q_1 = q_a$ over the interval $[t_1, t_2)$, the dynamics for $x_0^{ex,q_a} = [x_0^T, \bar{x}_a^T]^T$ is determined from (18) with

$$\mathbb{A}_0^{q_a} = \begin{bmatrix} A_0 & E_0 \\ \bar{G}_a & \bar{A}_a \end{bmatrix}, \quad \mathbb{M}_0^{q_a} = \begin{bmatrix} 0_{3\times1} \\ \bar{m}_a \end{bmatrix},$$

$$\mathbb{B}_0^{q_a} = \begin{bmatrix} B_0 \\ 0_{3\times1} \end{bmatrix}, \quad \mathbb{D}_0^{q_a} = \begin{bmatrix} D_0 & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}.$$

and $\mathbb{P}_0^{q_a}$ is given by

$$\mathbb{P}_0^{q_a} = [I_{3\times3}, 0_{3\times3}]^T P_0 [I_{3\times3}, 0_{3\times3}].$$

In this case, the values of the continuous state before and after $t_1$ are related by the jump map

$$x_0^{ex,q_a}(t_1) = \Psi_{0,a} x_0^{ex,q_0}(t_1-) \quad (20)$$

where

$$\Psi_{0,a} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} \end{bmatrix}. \quad (21)$$

In case (ii) where $q_1 = q_l$ holds, $x^{ex,q_l} = [x_0^T, \bar{x}_l^T]^T$ and

$$\mathbb{A}_0^{q_l} = \begin{bmatrix} A_0 & E_0 \\ \bar{G}_l & \bar{A}_l \end{bmatrix}, \quad \mathbb{M}_0^{q_l} = \begin{bmatrix} 0_{3\times1} \\ \bar{m}_l \end{bmatrix},$$

$$\mathbb{B}_0^{q_l} = \begin{bmatrix} B_0 \\ 0_{3\times1} \end{bmatrix}, \quad \mathbb{D}_0^{q_l} = \begin{bmatrix} D_0 & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}.$$

$$\mathbb{P}_0^{q_l} = [I_{3\times3}, 0_{3\times3}]^T P_0 [I_{3\times3}, 0_{3\times3}, 0_{3\times3}].$$

In this case, the values of the continuous state of the major trader before and after $t_1$ are related by the jump map

$$x_0^{ex,q_l}(t_1) = \Psi_{0,l} x_0^{ex,q_0}(t_1-) \quad (22)$$

where

$$\Psi_{0,l} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_{3\times3} \end{bmatrix}. \quad (23)$$

For the discrete state $q_2$, the continuous state of the major trader is $x_0^{ex,q_a} \equiv x_0$, and

$$\mathbb{A}_0^{q_2} = A_0, \quad \mathbb{M}_0^{q_2} = 0_{3\times1}, \quad \mathbb{B}_0^{q_2} = B_0, \quad \mathbb{D}_0^{q_2} = D_0$$

$$\bar{\mathbb{P}}_0^{q_2} = \bar{P}_0, \quad \mathbb{P}_0^{q_2} = P_0$$

The values continuous state of the major trader before and after $t_2$ are related by the the jump map

$$x_0^{ex,q_2}(t_2) = \Psi_{0,2} x_0^{ex,q_1}(t_2-) \tag{24}$$

where $\Psi_{0,2} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \end{bmatrix}$.

*Best response hybrid control action*:
By the definition of the terms $\mathbb{D}_0^{q_j}$ necessarily satisfy the condition A1 in [30], which in the LQG takes the following form

$$\mathbb{D}_0^{q_j} = \Psi_{0,j}\mathbb{D}_0^{q_{j-1}}, \qquad j = 1, 2. \tag{25}$$

An application of the stochastic hybrid control theory of [30], specialized to the LQG case in [31], yield the infinite population best response hybrid control action as

$$u_0^{q_j}(t) = -R_{0,q_j}^{-1}\mathbb{B}_{0,q_j}^T \Pi_0^{q_j}(t)\, x_0^{ex,q_j}(t), \tag{26}$$

where $\Pi_0^{q_j}(t)$ is the solution of

$$-\dot{\Pi}_0^{q_j} = \Pi_0^{q_j}\mathbb{A}_0^{q_j} + \mathbb{A}_{0,q_j}^T\Pi_0^{q_j} - \Pi_0^{q_j}\mathbb{B}_0^{q_j}R_{0,q_j}^{-1}\mathbb{B}_{0,q_j}^T\Pi_0^{q_j} + \mathbb{P}_0, \tag{27}$$

subject to the terminal and boundary conditions

$$\Pi_0^{q_2}(T) = \bar{\mathbb{P}}_0, \tag{28}$$
$$\Pi_0^{q_{j-1}}(t_j) = \Psi_{0,j}^T\Pi_0^{q_j}(t_j)\Psi_{0,j}, \tag{29}$$

$$\begin{aligned} &\mathbb{P}_0^{q_{j-1}} + \Psi_{0,j}^T\Pi_0^{q_j}(t_j)\Psi_{0,j}\mathbb{A}_0^{q_{j-1}} + \mathbb{A}_{0,q_{j-1}}^T\Psi_{0,j}^T\Pi_0^{q_j}(t_j)\Psi_{0,j} \\ &- \Psi_{0,j}^T\Pi_0^{q_j}(t_j)\Psi_{0,j}\mathbb{B}_0^{q_{j-1}}R_{0,q_{j-1}}^{-1}\mathbb{B}_{0,q_{j-1}}^T\Psi_{0,j}^T\Pi_0^{q_j}(t_j)\Psi_{0,j} \\ &\quad = \Psi_{0,j}^T\Big(\mathbb{P}_0^{q_j} + \Pi_0^{q_j}(t_j+)\mathbb{A}_0^{q_j} + \mathbb{A}_{0,q_j}^T\Pi_0^{q_j} \\ &- \Pi_0^{q_j}(t_j)\mathbb{B}_0^{q_j}R_{0,q_j}^{-1}\mathbb{B}_{0,q_j}^T\Pi_0^{q_j}(t_j)\Big)\Psi_{0,j}, \text{ for } j = 1, 2 \end{aligned} \tag{30}$$

We remark that $q_1 = q_a$ if $t_2^l < t_2^a$, where $t_2^a$ denotes the time $t_j,\ j = 2$ calculated from (30) with the substitution of $q_1 = q_a$, and $t_2^l$ denotes the time calculated with the substitution of $q_1 = q_l$. Otherwise, $q_1 = q_l$.

*2) Minor Acquirer:* A generic minor agent $\mathcal{A}_i$'s extended dynamics in the acquirer population with the extended state $x_i^{ex,q_j}$ is

$$dx_i^{ex,q_j} = (\mathbb{A}_a^{q_j}x_i^{ex,q_j} + \mathbb{M}_a^{q_j} + \mathbb{B}_0^{q_j}u_0^{q_i} + \mathbb{B}_a^{q_j}u_i^{q_i})dt + \mathbb{D}_a^{q_j}dW_i, \tag{31}$$

where for $q_0$, $x_i^{ex,q_0} = [x_i^T, x_0^T, \bar{x}_a^T, \bar{x}_l^T]^T$, and

$$\mathbb{A}_a^{q_0} = \begin{bmatrix} A_a & [G_a, E_a, E_a] \\ 0_{9\times3} & \mathbb{A}_0^{q_0} - \mathbb{B}_0^{q_0}R_{0,q_0}^{-1}\mathbb{B}_{0,q_0}^T\Pi_0^{q_0} \end{bmatrix},$$

$$\mathbb{M}_a^{q_0} = \begin{bmatrix} 0_{3\times1}, \\ \mathbb{M}_0 \end{bmatrix},\ \mathbb{B}_a^{q_0} = \begin{bmatrix} B_a \\ 0_{9\times1} \end{bmatrix},\ \mathbb{D}_a^{q_0} = \begin{bmatrix} D_a & 0_{3\times9} \\ 0_{9\times3} & \mathbb{D}_0^{q_0} \end{bmatrix},$$

and for $q_a$, $x_i^{ex,q_a} = [x_i^T, x_0^T, \bar{x}_a^T]^T$, and

$$\mathbb{A}_a^{q_a} = \begin{bmatrix} A_a & [G_a, E_a] \\ 0_{6\times3} & \mathbb{A}_0^{q_a} - \mathbb{B}_0^{q_a}R_{0,q_a}^{-1}\mathbb{B}_{0,q_a}^T\Pi_0^{q_a} \end{bmatrix},$$

$$\mathbb{M}_a^{q_a} = \begin{bmatrix} 0_{3\times1} \\ \mathbb{M}_0 \end{bmatrix},\ \mathbb{B}_a^{q_a} = \begin{bmatrix} B_a \\ 0_{6\times1} \end{bmatrix},\ \mathbb{D}_a^{q_a} = \begin{bmatrix} D_a & 0_{3\times6} \\ 0_{6\times3} & \mathbb{D}_0^{q_a} \end{bmatrix}.$$

In case (i) where the acquirer population is trading over $[t_1, t_2)$, i.e. $q_1 = q_a$, the total hybrid cost for a minor acquirer is given by

$$\begin{aligned} J_i^a(u_i, u_{-i}) = \mathbb{E}\Big[&\|x_i^{ex,q_a}(t_2)\|^2_{\bar{\mathbb{P}}_a^{q_a}} \\ &+ \sum_{j=0}^{1}\int_{t_j}^{t_{j+1}}\big(\|x_i^{ex,q_j}(s)\|^2_{\mathbb{P}_a^{q_j}} + \|u_i^{q_j}(s)\|^2_{R_a^{q_j}}\big)ds\Big], \end{aligned} \tag{32}$$

with

$$\bar{\mathbb{P}}_a^{q_a} = [I_{3\times3}, 0_{3\times6}]^T\bar{P}_a[I_{3\times3}, 0_{3\times6}] \tag{33}$$
$$\mathbb{P}_a^{q_a} = [I_{3\times3}, 0_{3\times6}]^T P_a[I_{3\times3}, 0_{3\times6}] \tag{34}$$
$$\mathbb{P}_a^{q_0} = [I_{3\times3}, 0_{3\times9}]^T P_a[I_{3\times3}, 0_{3\times9}], \tag{35}$$

where $\bar{P}_a, P_a$ are, respectively, associated with the running and final costs in (7) are given by

$$\bar{P}_a = \begin{bmatrix} \mu_a & -\frac{1}{2}p_a a & 0 \\ -\frac{1}{2}p_a a & p_a\psi_a & \frac{1}{2}p_a \\ 0 & \frac{1}{2}p_a & \xi_a \end{bmatrix}, P_a = \begin{bmatrix} \varrho_a & 0 & -\frac{1}{2}r_a \\ 0 & \kappa_a & 0 \\ -\frac{1}{2}r_a & 0 & \gamma_a \end{bmatrix}.$$

In this case, the extended state for a generic minor agent in the acquirer population at $t_1$ satisfies the jump transition map

$$x^{ex,q_a}(t_1) = \Psi_{i,a}x^{ex,q_0}(t_1-)$$

with

$$\Psi_{i,a} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} \end{bmatrix}.$$

In case (ii) where $q_1 = q_l$ holds over the interval $[t_1, t_2)$, the cost for the minor acquirer agent $\mathcal{A}_i$ is given by

$$\begin{aligned} J_i^a(u_i, u_{-i}) = \mathbb{E}\Big[&\|x_i^{ex,q_0}(t_1)\|^2_{\bar{\mathbb{P}}^{q_0}} \\ &+ \int_{t_0}^{t_1}\big(\|x_i^{ex,q_0}(s)\|^2_{\mathbb{P}_a^{q_0}} + \|u_i^{q_j}(s)\|^2_{R_a^{q_0}}\big)ds\Big], \end{aligned} \tag{36}$$

with $\bar{\mathbb{P}}^{q_0} = [I_{3\times3}, 0_{3\times9}]^T\bar{P}[I_{3\times3}, 0_{3\times9}]$.

*Best response hybrid control action*:
The optimal stopping problem for a minor acquirer is equivalent to a hybrid optimal control problem in which the dynamics and costs become zero after stopping. By the definition of the terms $\mathbb{D}_a^{q_j}$ necessarily satisfy the condition A1 in [30]. To be specific, for the case (i) the diffusion coefficients in (31) satisfy

$$\mathbb{D}_a^{q_a} = \Psi_{i,a}\mathbb{D}_a^{q_0}, \tag{37}$$
$$\mathbb{D}_a^{q_2} = \Psi_{i,\sigma_{q_a,q_{\text{stop}}}}\mathbb{D}_a^{q_a} \equiv 0, \tag{38}$$

where $\sigma_{q_j,q_{\text{stop}}}$ denotes the stopping event in the discrete state $q_j$. Both conditions in (38) are satisfied since $\mathbb{D}_a^{q_2} = 0$ due to the zero dynamics after stopping and $\Psi_{i,\sigma_{q_a,q_{\text{stop}}}} = 0$ due to removal of the minor acquirer trader's state from the market dynamics. For the case (ii) we also have

$$\mathbb{D}_a^{q_1} = \Psi_{i,\sigma_{q_0,q_{\text{stop}}}}\mathbb{D}_a^{q_0} \equiv 0, \tag{39}$$

which holds due to the stopping decision at $t_1$. The results of [30], [31] yield

$$u_i^{q_j}(t) = -R_{q_j}^{-1}\mathbb{B}_{a,q_j}^T\Pi_a^{q_j}(t)\, x_i^{ex,q_j}(t), \tag{40}$$

with

$$-\dot{\Pi}_a^{q_j} = \Pi_a^{q_j}\mathbb{A}_a^{q_j} + \mathbb{A}_{a,q_j}^T\Pi_a^{q_j} - \Pi_a^{q_j}\mathbb{B}_a^{q_j}R_{a,q_j}^{-1}\mathbb{B}_{a,q_j}^T\Pi_a^{q_j} + \mathbb{P}_a, \quad (41)$$

where for the case (i), in which $q_1 = q_a$, $\Pi_a^{q_j}(t)$ is the solution of (41) subject to the terminal conditions

$$\Pi_a^{q_a}(t_2) = \bar{\mathbb{P}}_a^{q_a},$$

$$\Big(\mathbb{P}_a^{q_a} + \bar{\mathbb{P}}_a^{q_a}\mathbb{A}_a^{q_a} + \mathbb{A}_{a,q_a}^T\bar{\mathbb{P}}_a^{q_a} - \bar{\mathbb{P}}_a^{q_a}\mathbb{B}_a^{q_a}R_{a,q_a}^{-1}\mathbb{B}_{a,q_a}^T\bar{\mathbb{P}}_a^{q_a}\Big)_{t=t_2} = 0$$

and the boundary conditions

$$\begin{aligned}\Pi_a^{q_0}(t_1) &= \Psi_{i,a}^T\Pi_a^{q_a}(t_1)\Psi_{i,a},\\ \mathbb{P}_a^{q_0} + \Psi_{i,a}^T\Pi_a^{q_a}(t_1)\Psi_{i,a}\mathbb{A}_a^{q_0} &+ \mathbb{A}_{a,q_0}^T\Psi_{i,a}^T\Pi_a^{q_a}(t_1)\Psi_{i,a}\\ - \Psi_{i,a}^T\Pi_a^{q_a}(t_1)\Psi_{i,a}\mathbb{B}_a^{q_0}&R_{a,q_0}^{-1}\mathbb{B}_{a,q_0}^T\Psi_{i,a}^T\Pi_a^{q_a}(t_1)\Psi_{i,a}\\ = \Psi_{i,a}^T\Big(\mathbb{P}_a^{q_a} &+ \Pi_a^{q_a}(t_1)\mathbb{A}_a^{q_a} + \mathbb{A}_{a,q_a}^T\Pi_a^{q_a}(t_1)\\ &- \Pi_a^{q_a}(t_1)\mathbb{B}_a^{q_a}R_{a,q_a}^{-1}\mathbb{B}_{a,q_a}^T\Pi_a^{q_a}(t_1)\Big)\Psi_{i,a}, \quad (42)\end{aligned}$$

and in case (ii) where $q_1 = q_l$ holds, $\Pi_a^{q_0}(t)$ is the solution of (41) subject to the terminal conditions

$$\Pi_a^{q_0}(t_1) = \bar{\mathbb{P}}_a^{q_0} \quad (43)$$

$$\Big(\mathbb{P}_a^{q_0} + \bar{\mathbb{P}}_a^{q_0}\mathbb{A}_a^{q_0} + \mathbb{A}_{a,q_0}^T\bar{\mathbb{P}}_a^{q_0} - \bar{\mathbb{P}}_a^{q_0}\mathbb{B}_a^{q_0}R_{a,q_0}^{-1}\mathbb{B}_{a,q_0}^T\bar{\mathbb{P}}_a^{q_0}\Big)_{t=t_1} = 0. \quad (44)$$

We remark that the the solution of (42) (respectively (44)) result in the same stopping times $t_2^a$ (respectively $t_1$) determined from (30).

*3) Minor Liquidator:* The hybrid dynamics, jump maps and performance measures for a minor liquidator are presented in a similar form as the minor acquirer, and therefore, due to space limitations, are not presented here. The infinite population best response hybrid control action as

$$u_i^{q_j}(t) = -R_{q_j}^{-1}\mathbb{B}_{l,q_j}^T\Pi_l^{q_j}(t)\, x_i^{ex,q_j}(t), \quad (45)$$

with

$$-\dot{\Pi}_l^{q_j} = \Pi_l^{q_j}\mathbb{A}_l^{q_j} + \mathbb{A}_{l,q_j}^T\Pi_l^{q_j} - \Pi_l^{q_j}\mathbb{B}_l^{q_j}R_{l,q_j}^{-1}\mathbb{B}_{l,q_j}^T\Pi_l^{q_j} + \mathbb{P}_l, \quad (46)$$

where for the case (i), in which $q_1 = q_a$, $\Pi_l^{q_j}(t)$ is the solution of (46) subject to the terminal conditions

$$\Pi_l^{q_0}(t_1) = \bar{\mathbb{P}}_l^{q_0}$$

$$\Big(\mathbb{P}_l^{q_0} + \bar{\mathbb{P}}_l^{q_0}\mathbb{A}_l^{q_0} + \mathbb{A}_{l,q_0}^T\bar{\mathbb{P}}_l^{q_0} - \bar{\mathbb{P}}_l^{q_0}\mathbb{B}_l^{q_0}R_{l,q_0}^{-1}\mathbb{B}_{l,q_0}^T\bar{\mathbb{P}}_l^{q_0}\Big)_{t=t_1} = 0.$$

and in case (ii) where $q_1 = q_l$ holds, $\Pi_l^{q_0}(t)$ is the solution of (41) subject to the terminal conditions

$$\Pi_l^{q_l}(t_2) = \bar{\mathbb{P}}_l^{q_l},$$

$$\Big(\mathbb{P}_l^{q_l} + \bar{\mathbb{P}}_l^{q_l}\mathbb{A}_l^{q_l} + \mathbb{A}_{l,q_l}^T\bar{\mathbb{P}}_l^{q_l} - \bar{\mathbb{P}}_l^{q_l}\mathbb{B}_l^{q_l}R_{l,q_l}^{-1}\mathbb{B}_{l,q_l}^T\bar{\mathbb{P}}_l^{q_l}\Big)_{t=t_2} = 0$$

and the boundary conditions

$$\begin{aligned}\Pi_l^{q_0}(t_1) &= \Psi_{i,l}^T\Pi_l^{q_l}(t_1)\Psi_{i,l},\\ \mathbb{P}_l^{q_0} + \Psi_{i,l}^T\Pi_l^{q_l}(t_1)\Psi_{i,l}\mathbb{A}_l^{q_0} &+ \mathbb{A}_{l,q_0}^T\Psi_{i,l}^T\Pi_l^{q_l}(t_1)\Psi_{i,l}\\ - \Psi_{i,l}^T\Pi_l^{q_l}(t_1)\Psi_{i,l}\mathbb{B}_l^{q_0}&R_{l,q_0}^{-1}\mathbb{B}_{l,q_0}^T\Psi_{i,l}^T\Pi_l^{q_l}(t_1)\Psi_{i,l}\\ = \Psi_{i,l}^T\Big(\mathbb{P}_l^{q_l} &+ \Pi_l^{q_l}(t_1)\mathbb{A}_l^{q_l} + \mathbb{A}_{l,q_l}^T\Pi_l^{q_l}(t_1)\\ &- \Pi_l^{q_l}(t_1)\mathbb{B}_l^{q_l}R_{l,q_l}^{-1}\mathbb{B}_{l,q_l}^T\Pi_l^{q_l}(t_1)\Big)\Psi_{i,l}. \quad (47)\end{aligned}$$

The infinite population equilibria is linked to the finite population equilibria by the following theorem.

*Theorem 3.1 ($\epsilon$-Nash Equilibria for CO MM-MF Systems):* Subject to reasonable technical conditions (see [7]), the system equations (8), (10), (11) together with the mean field equations (49) generate the set of control laws $\mathcal{U}_{MF}^N \triangleq \{u_i^\circ; 0 \le i \le N\}$, $1 \le N < \infty$, given by (26), (40), and (45) such that

(i) All agent systems $\mathcal{A}_i$, $0 \le i \le N$, are second order stable.

(ii) $\{\mathcal{U}_{MF}^N; 1 \le N < \infty\}$ yields an $\epsilon$-Nash equilibrium for all $\epsilon$, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \ge N(\epsilon)$;

$$J_i^{s,N}(u_i^\circ, u_{-i}^\circ) - \epsilon \le \inf_{u_i \in \mathcal{U}_{i,y}^N} J_i^{s,N}(u_i, u_{-i}^\circ) \le J_i^{s,N}(u_i^\circ, u_{-i}^\circ).$$

□

## E. Consistency Condition

The closed loop trading dynamics of a minor acquirer $\mathcal{A}_i \in \mathcal{A}_a$ applying (40), or correspondingly a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$ applying (45) is consequently

$$d\nu_i = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T dt - R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}(t)dt,$$

then the average of closed loop trading dynamics over acquirer or liquidator population is obtained as

$$\begin{aligned}\frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} d\nu_i = -\frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T dt\\ - \frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}(t)dt, \quad (48)\end{aligned}$$

where $\bar{x} = [\bar{x}_a^T, \bar{x}_l^T]^T$. Then taking the $L^2$ limit of (48) as the population size $N_{a/l}$ goes to infinity yields the trading rate mean field dynamics

$$\begin{aligned}d\bar{\nu}_{a/l} &= \lim_{N_{a/l}\to\infty} d\nu^{N_{a/l}} = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l}\\ &\times \lim_{N_{a/l}\to\infty}\big((x^{N_{a/l}})^T, x_0^T, \bar{x}_a^T, \bar{x}_l^T\big)^T dt - R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}dt,\end{aligned}$$

and hence the consistency equations become

$$\begin{aligned}\bar{A}_{a,11} &= -R_a^{-1}(\Pi_{a,11} + \Pi_{a,17}) - aR_a^{-1}(\Pi_{a,31} + \Pi_{a,37}),\\ \bar{A}_{a,12} &= -R_a^{-1}(\Pi_{a,12} + \Pi_{a,18}) - aR_a^{-1}(\Pi_{a,32} + \Pi_{a,38}),\\ \bar{A}_{a,13} &= -R_a^{-1}(\Pi_{a,13} + \Pi_{a,19}) - aR_a^{-1}(\Pi_{a,33} + \Pi_{a,39})\\ ,\bar{A}_{al,11} &= -R_a^{-1}(\Pi_{a,110} + a\Pi_{a,310}),\\ \bar{A}_{al,12} &= -R_a^{-1}(\Pi_{a,111} + a\Pi_{a,311})\\ \bar{A}_{al,13} &= -R_a^{-1}(\Pi_{a,112} + a\Pi_{a,312})\\ \bar{G}_{a/l,11} &= -R_{a/l}^{-1}(\Pi_{a/l,14} + a\Pi_{a/l,34}),\\ \bar{G}_{a/l,12} &= -R_{a/l}^{-1}(\Pi_{a/l,15} + a\Pi_{a/l,35}),\\ \bar{G}_{a/l,13} &= -R_{a/l}^{-1}(\Pi_{a/l,16} + a\Pi_{a/l,36}),\\ \bar{m}_{a/l,1} &= 0, \quad (49)\end{aligned}$$

where $\Pi_{a/l,ij} = \Pi_{a/l}(i,j)$ for $i = \{1,3\}$, $j = \{1,2,3,...,12\}$. Hence the matrices in (15) are given as

$$\bar{A}_{a/l} = \begin{bmatrix} \bar{A}_{a/l,11} & \bar{A}_{a/l,12} & \bar{A}_{a/l,13} \\ 1 & 0 & 0 \\ (\lambda + a\bar{A}_{a/l,11}) & a\bar{A}_{a/l,12} & a\bar{A}_{a/l,13} \end{bmatrix},$$

$$\bar{A}_{al}=\begin{bmatrix} \bar{A}_{al,11} & \bar{A}_{al,12} & \bar{A}_{al,13} \\ 0 & 0 & 0 \\ a\bar{A}_{al,11} & a\bar{A}_{al,12} & a\bar{A}_{al,13} \end{bmatrix}, \ \bar{m}=\begin{bmatrix} \bar{m}_{a/l,1} \\ 0 \\ a\bar{m}_{a/l,1} \end{bmatrix}$$

$$\bar{A}_{la} = \begin{bmatrix} \bar{A}_{la,11} & \bar{A}_{la,12} & \bar{A}_{la,13} \\ 0 & 0 & 0 \\ a\bar{A}_{la,11} & a\bar{A}_{la,12} & a\bar{A}_{la,13} \end{bmatrix},$$

$$\bar{G}_{a/l} = \begin{bmatrix} \bar{G}_{a/l,12} & \bar{G}_{a/l,22} & \bar{G}_{a/l,23} \\ 0 & 0 & 0 \\ (\lambda_0 + a\bar{G}_{a/l,21}) & a\bar{G}_{a/l,22} & a\bar{G}_{a/l,23} \end{bmatrix}.$$

where the equations for the entries of matrices $\bar{A}_{la}, \bar{A}_l$ due to space limitations are not presented (see [41]).

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