On the Stochastic Minimum Principle for Hybrid Systems

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***Abstract*— A class of stochastic hybrid systems with both autonomous and controlled switchings and jumps is considered where autonomous and controlled state jumps at the switching instants are accompanied by changes in the dimension of the state space. Optimal control problems associated with this class of stochastic hybrid systems are studied where in addition to running and terminal costs, switching between discrete states incurs costs. Necessary optimality conditions are established in the form of the Stochastic Hybrid Minimum Principle. A feature of special importance is the effect of hard constraints imposed by switching manifolds on diffusion-driven state trajectories which influence the boundary conditions for the stochastic Hamiltonian and adjoint processes.**

I. Introduction

The Minimum Principle (MP), also called the Maximum Principle in the pioneering work of Pontryagin et al. [1], is a milestone of systems and control theory that led to the emergence of optimal control as a distinct field of research. This principle states that any optimal control along with the optimal state trajectory must solve a two-point boundary value problem in the form of an extended Hamiltonian canonical system, as well as a minimization condition (or a maximization, depending on the sign convention used) for the Hamiltonian function. Since the original publication of the MP [1], which was established for deterministic and continuous dynamical systems, there has been a considerable effort for the generalization of the Minimum Principle for broader classes of control systems.

The generalization of the Minimum Principle for hybrid systems, i.e. control systems with both continuous and discrete states and dynamics, results in the Hybrid Minimum Principle (HMP) (see e.g. [2]–[15]). The HMP gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system with fixed initial conditions and a sequence of autonomous and controlled switchings. These conditions are expressed in terms of the minimization of the distinct Hamiltonians indexed by the discrete state sequence of the hybrid trajectory. A feature of special interest is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations [16].

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The generalization of the Minimum Principle for continuous parameter stochastic systems results in the Stochastic Minimum Principle (SMP) (see e.g. [17]–[24]). When diffusion terms are functions of the system state only, the SMP is derived via similar first-order variational analyses as those employed in the derivation of the deterministic MP. However, unlike the deterministic case for which backward ordinary differential equations for the adjoint process are equivalent to a forward ODE with a reversal of time, the backward stochastic differential equation for the adjoint process must remain non-anticipative, requiring the solution to be $\mathfrak{I}^t$-adapted. When diffusion terms also depend on the controls, one is required to study both the first-order and second-order variations and derive the SMP using a stochastic Hamiltonian system consisting of two forward-backward stochastic differential equations and a minimization condition with an additional term quadratic in the diffusion coefficient (see e.g. [19]–[21]).

The optimal control of stochastic hybrid systems, i.e. control systems that involve the interaction of continuous dynamics, discrete dynamics and stochastic diffusions, has been the subject of a limited number of studies. The SMP formulation in [25] considers only controlled switching and jumps, and the Stochatsic Dynamic Programming (SDP) formulation in [26] studies infinite horizon problems where optimal controls are stationary. In this paper, the hybrid systems framework in [12]–[14], [27]–[29] is extended to cover a general class of stochastic hybrid systems with state dependant diffusion fields which are subject to autonomous and controlled switchings and state jumps. A feature of special interest is the effect of hard constraints imposed by switching manifolds on diffusion-driven state trajectories, that to the best of our knowledge has not been considered in the literature before. Furthermore, autonomous and controlled state jumps at switching instants are allowed to be accompanied by changes in the dimension of the state space. Optimal control problems for such stochastic hybrid systems are studied in the presence of a large range of running, terminal and switching costs. First order variational analysis is performed on the stochastic hybrid optimal control problem via the needle variation methodology and the necessary optimality conditions are established in the form of the Stochastic Hybrid Minimum Principle (SHMP).

II. Basic Assumptions

Let $(\Omega, \mathfrak{I}, P)$ be a probability space with filtration $\mathfrak{I}^t$, let $w(.)$ be a standard $\mathbb{R}^{n_w}$ valued Wiener process. Consider a *hybrid system* $\mathbb{H}$ as an octuple

$$\mathbb{H} = \{H := Q \times M, I := \Sigma \times U, \Gamma, A, F, G, \Xi, \mathscr{M}\}, \quad (1)$$

where the symbols in the expression and their governing assumptions are defined as below.

A0: $\Im^t = \boldsymbol{\sigma}\{w(s) : 0 \le s \le t\}$, where $\boldsymbol{\sigma}$ denotes sigma-algebra.

$H := Q \times M$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$, where

$Q = \{1, 2, ..., |Q|\} \equiv \{q_1, q_2, ..., q_{|Q|}\}, |Q| < \infty$, is a finite set of *discrete states (components)*, and

$M = \{\mathbb{R}^{n_q}\}_{q \in Q}$ is a family of finite dimensional continuous valued state spaces, where $n_q \le n < \infty$ for all $q \in Q$.

$I := \Sigma \times U$ is the set of system input values, where

Σ with $|\Sigma| < \infty$ is the set of discrete state transition and continuous state jump events extended with the identity element, and

$U = \{U_q\}_{q \in Q}$ is the set of *admissible input control values*, where each $U_q \subset \mathbb{R}^{m_q}$ is a compact set in $\mathbb{R}^{m_q}$.

The set of admissible (continuous) control inputs $\mathscr{U}(U) := L_\infty([t_0, T_*), U)$, is defined to be the set of $\Im^t$-adapted measurable functions that are bounded up to a set of measure zero on $[t_0, T_*), T_* < \infty$. The boundedness property necessarily holds since admissible input functions take values in the compact set U.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete state transition map*.

$\Xi : H \times \Sigma \to H$ is a time independent (partially defined) *continuous state jump transition map*. All $\xi_\sigma \in \Xi$, $\xi_\sigma : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$, $p \in A(q, \sigma)$ are assumed to be continuously differentiable in the continuous state $x \in \mathbb{R}^{n_q}$. In this paper, we only consider linear jump maps for which continuous differentiability automatically holds and further, $\xi(c_1 x_1 + c_2 x_2) = c_1 \xi(x_1) + c_2 \xi(x_2) \equiv c_1 \nabla \xi\, x_1 + c_2 \nabla \xi\, x_2$ for $c_1, c_2 \in \mathbb{R}$, $x_1, x_2 \in \mathbb{R}^n$.

$A : Q \times \Sigma \to Q$ denotes both a finite automaton and the automaton's associated transition function on the state space Q and event set Σ, such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q-dependent subset $\{A(q, \sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : Q \times \mathbb{R}^n \times \Sigma \to H|_Q$. In other words, Γ can only make a discrete state transition in a hybrid state (q, x) if the automaton A can make the corresponding transition in q.

F is an indexed collection of Borel measurable *vector fields* $\{f_q\}_{q \in Q}$ such that $f_q \in C^{k_{f_q}}(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}^{n_q})$, $k_{f_q} \ge 1$, satisfies a joint uniform boundedness and Lipschitz condition, i.e. there exists $L_f < \infty$ such that $\|f_q(x, u)\| \le L_f(1 + \|x\| + \|u\|)$ and $\|f_q(x_1, u_1) - f_q(x_2, u_2)\| \le L_f(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $x, x_1, x_2 \in \mathbb{R}^{n_q}$, $u, u_1, u_2 \in U_q$, $q \in Q$.

G is an indexed collection of Borel measurable *diffusion fields* $\{g_q\}_{q \in Q}$ such that $g_q \in C^{k_{g_q}}(\mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times n_w})$, $k_{g_q} \ge 1$, satisfies a uniform boundedness and Lipschitz condition, i.e. there exists $L_g < \infty$ such that $\|g_q(x)\| \le L_g(1 + \|x\|)$ and $\|g_q(x_1) - g_q(x_2)\| \le L_g \|x_1 - x_2\|$, for all $x_1, x_2 \in \mathbb{R}^{n_q}$, $q \in Q$.

$\mathscr{M} = \{m_\alpha : \alpha \in Q \times Q,\}$ denotes a collection of *switching manifolds* such that, for any ordered pair $\alpha \equiv (\alpha_1, \alpha_2) = (q, r)$, m_α is a smooth, i.e. C^∞, codimension 1 sub-manifold of $\mathbb{R}^{n_q}$, described locally by $m_\alpha = \{x \in \mathbb{R}^{n_q} : m_\alpha(x) = 0\}$. It is assumed that $m_\alpha \cap m_\beta = \emptyset$, whenever $\alpha_1 = \beta_1$ but $\alpha_2 \ne \beta_2$, for all $\alpha, \beta \in Q \times Q$. □

We note that the case where m_α is identified with its reverse ordered version $m_{\bar{\alpha}}$ giving $m_\alpha = m_{\bar{\alpha}}$ is not ruled out by this definition, even in the non-trivial case $m_{p,p}$ where $\alpha_1 = \alpha_2 = p$. The former case corresponds to the common situation where the switching of vector fields at the passage of the continuous trajectory in one direction through a switching manifold is reversed if a reverse passage is performed by the continuous trajectory, while the latter case corresponds to the standard example of the bouncing ball.

Switching manifolds will function in such a way that whenever a trajectory governed by the controlled vector field and the diffusion field meets the switching manifold transversally there is an autonomous switching to another controlled vector field or there is a jump transition in the continuous state component, or both. A transversal arrival on a switching manifold $m_{q,r}$, at state x occurs whenever

$$\nabla m_{q,r}(x)^T f_q(x, u) \ne 0, \quad (2)$$

for $x \in \{x \in \mathbb{R}^{n_q} : m_{q,r}(x) = 0\}$, $u \in U_q$, $q, r \in Q$.

A1: In this paper, we further assume that

$$g_r\left(\xi_{\sigma_{q,r}}(x)\right) = \xi_{\sigma_{q,r}}\left(g_q(x)\right), \quad (3)$$

and for all $x \in \{x \in \mathbb{R}^{n_q} : m_{q,r}(x) = 0\}$ we assume that

$$\langle g_q(x), \nabla m_{q,r}(x)\rangle = 0. \quad (4)$$

□

The former condition considers equivalent diffusion fields before and after switching events and the latter corresponds to the absence of transversal diffusion fields on the switching surface. For the case of systems under turbulence-driven diffusion fields and with switching manifolds formed by solid surfaces both (3) and (4) in A1 automatically hold. In addition to the basic assumptions in A0 and A1, it is assumed that:

A2: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0,q}(x_0) \ne 0$, for all $q \in Q$. □

III. Hybrid Optimal Control Problems

A3: Let $\{l_q\}_{q \in Q}, l_q \in C^{n_l}(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}_+), n_l \ge 1$, be a family of Borel measurable running cost functions; $\{c_\sigma\}_{\sigma \in \Sigma} \in C^{n_c}(\mathbb{R}^{n_q} \times \Sigma \to \mathbb{R}_+), n_c \ge 1$, be a family of Borel measurable switching cost functions; and $h \in C^{n_h}\left(\mathbb{R}^{n_{q_f}} \to \mathbb{R}_+\right), n_h \ge 1$, be a Borel measurable terminal cost function satisfying the following assumptions:

(i) There exists $K_l < \infty$ and $1 \le \gamma_l < \infty$ such that $|l_q(x_1, u_1) - l_q(x_2, u_2)| \le K_l(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $x_1, x_2 \in \mathbb{R}^{n_q}$, $u_1, u_2 \in U_q$, $q \in Q$.

(ii) There exists $K_c < \infty$ and $1 \le \gamma_c < \infty$ such that $|c_\sigma(x)| \le K_c\left(1 + \|x\|^{\gamma_c}\right)$, $\sigma \in \Sigma$, $x \in \mathbb{R}^{n_q}$, $q \in Q$.

(iii) There exists $K_h < \infty$ and $1 \le \gamma_h < \infty$ such that $|h(x)| \le K_h\left(1 + \|x\|^{\gamma_h}\right)$, $x \in \mathbb{R}^{n_{q_f}}$, $q_f \in Q$. □

Consider the initial time t_0, final time $t_f < \infty$, and initial hybrid state $h_0 = (q_0, x_0)$. For a fixed number of switchings $L < \infty$, let $\tau_L := \{t_0, t_1, t_2, \ldots, t_L\}$ be a strictly increasing $\Im^t$-adapted sequence of times and $\sigma_i \in \Sigma$, $i \in \{1, 2, \cdots, L\}$ extended with $\sigma_0 = id$ be a *discrete event sequence* that form a hybrid switching sequence

$$S_L = \left\{(t_0, id), \left(t_1, \sigma_{q_0 q_1}\right), \ldots, \left(t_L, \sigma_{q_{L-1} q_L}\right)\right\} \equiv \left\{(t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L)\right\}. \quad (5)$$

With the set of admissible continuous control inputs given as $\mathscr{U} = \bigcup_{i=0}^{L} L_\infty\left([t_i, t_{i+1}), U_{q_i}\right)$ with $t_{L+1} = t_f$, a $\Im^t$-adapted *hybrid input process* is denoted by $I_L := (S_L, u)$, $u \in \mathscr{U}$, $u(t) : \Im^t -$ measurable.

Consider the hybrid performance function

$$J\left(t_0, t_f, h_0, L; I_L\right) := \mathbb{E}\left\{\sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}(s), u(s)\right) ds + \sum_{j=1}^{L} c_{\sigma_{q_{j-1} q_j}}\left(t_j, x_{q_{j-1}}(t_j-)\right) + h\left(x_{q_L}(t_f)\right)\right\}, \quad (6)$$

subject to

$$dx_{q_i}(t) = f_{q_i}(x_{q_i}(t), u_{q_i}(t))\, dt + g_{q_i}(x_{q_i}(t))\, dw, \quad t \in [t_i, t_{i+1}), \quad (7)$$

$$x_{q_0}(t_0) = x_0, \quad (8)$$

$$x_{q_j}(t_j) = \xi_{\sigma_{q_{j-1} q_j}}\left(x_{q_{j-1}}(t_j-)\right) \equiv \xi_{\sigma_{q_{j-1} q_j}}\left(\lim_{t \uparrow t_j} x_{q_{j-1}}(t)\right), \quad (9)$$

where $0 \le i \le L$, $1 \le j \le L$, $t_{L+1} = t_f < \infty$. If t_j is the time of an autonomous switching, then

$$m_{q_{j-1} q_j}\left(x_{q_{j-1}}(t_j-)\right) = 0. \quad (10)$$

The Hybrid Optimal Control Problem (HOCP) is defined as the infimization of the hybrid cost (6) over the family of hybrid input trajectories with L switchings $\boldsymbol{I}_L$, i.e.

$$J^o\left(t_0, t_f, h_0, L\right) = \inf_{I_L \in \boldsymbol{I}_L} J\left(t_0, t_f, h_0, L; I_L\right). \quad (11)$$

□

IV. Stochastic Hybrid Minimum Principle

Theorem 1 Consider the hybrid system $\mathbb{H}$ together with the assumptions A0, A1, A2 and A3 as above and the HOCP (11) for the hybrid cost (6). Define the family of system Hamiltonians by

$$H_q\left(x_q, u_q, \lambda_q, K_q\right) = l_q\left(x_q, u_q\right) + \lambda_q^T f_q\left(x_q, u_q\right) + \operatorname{tr}\left[K_q^T g_q\left(x_q\right)\right], \quad (12)$$

with $q \in Q$, $x_q \in \mathbb{R}^{n_q}$, $u_q \in U_q$, $\lambda_q \in \mathbb{R}^{n_q}$, $K_q \in \mathbb{R}^{n_q \times n_w}$. Then for the optimal input u^o and the corresponding trajectory x^o there exists $\lambda^o, K_q^o : \Im^t -$ adapted, such that

$$d x_q{}^o = \frac{\partial H_{q^o}}{\partial \lambda_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dt + \frac{\partial H_{q^o}}{\partial K_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dw, \quad (13)$$

$$d\lambda_q{}^o = -\frac{\partial H_{q^o}}{\partial x_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dt + K_q^o dw, \quad (14)$$

almost everywhere $t \in \left[t_0, t_f\right]$ with

$$x_{q_0}^o(t_0) = x_0, \quad (15)$$

$$x_{q_j}^o(t_j) = \xi_{\sigma_{q_{j-1}, q_j}}\left(x_{q_{j-1}}^o(t_j-)\right), \quad (16)$$

$$\lambda_{q_L}^o(t_f) = \frac{\partial g}{\partial x_{q_L}}\left(x_{q_L}^o(t_f)\right), \quad (17)$$

$$\lambda_{q_{j-1}}^o(t_j) = \left[\frac{\partial \xi_{\sigma_{q_{j-1}, q_j}}}{\partial x_{q_{j-1}}}\right]^T \lambda_{q_j}^o(t_j+) + p\,\frac{\partial m_{q_{j-1}, q_j}}{\partial x_{q_{j-1}}} + \frac{\partial c_{\sigma_{q_{j-1}, q_j}}}{\partial x_{q_{j-1}}}, \quad (18)$$

where $p \in \mathbb{R}$ when t_j indicates the time of an autonomous switching, and $p = 0$ when t_j indicates the time of a controlled switching.

Moreover,

$$H_{q^o}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) \le H_{q^o}\left(x_q^o, v, \lambda_q^o, K_q^o\right), \quad (19)$$

almost everywhere $t \in \left[t_0, t_f\right]$, almost surely for all $v : \Im^t -$ measurable random variables in U_q, that is to say the Hamiltonian is minimized with respect to the control input; and at a switching time t_j the Hamiltonian satisfies

$$H_{q_{j-1}}(t_j-) \equiv H_{q_{j-1}}(t_j) = H_{q_j}(t_j) \equiv H_{q_j}(t_j+). \quad (20)$$

□

Proof: This is a brief version of the proof in [30] to appear in detail in a consecutive paper. Consider the case of a hybrid optimal control problem with a single switching case, i.e. with $L = 1$, $t_f = t_{L+1} = t_2$ and with the notation $t_s := t_1$.

First, consider a needle variation at time $t \in \left(t_s, t_f\right)$ in the form of

$$u^\varepsilon(\tau) = \begin{cases} u_{q_0}^o(\tau) & \text{if} \quad t_0 \le \tau < t_s \\ u_{q_1}^o(\tau) & \text{if} \quad t_s \le \tau < t \\ v & \text{if} \quad t \le \tau < t + \varepsilon \\ u_{q_1}^o(\tau) & \text{if} \quad t + \varepsilon \le \tau \le t_f \end{cases}. \quad (21)$$

This corresponds to a perturbed trajectory $\hat{x}^\varepsilon(\tau), \tau \in \left[t_0, t_f\right]$ for which $x_{q_0}^\varepsilon(\tau) = x_{q_0}^o(\tau)$, $t_0 \le \tau < t_s$ and $x_{q_1}^\varepsilon(\tau) = x_{q_1}^o(\tau)$, $t_s \le \tau \le t$, and for $t \le \tau \le t_f$ we may write:

$$\begin{aligned} \delta x_{q_1}^\varepsilon(\tau) &:= x_{q_1}^\varepsilon(\tau) - x_{q_1}^o(\tau) \\ &= \int_t^{t+\varepsilon} \left[f_{q_1}\left(x_{q_1}^\varepsilon(s), v\right) - f_{q_1}\left(x_{q_1}^o(s), u_{q_1}^o(s)\right)\right] ds \\ &+ \int_{t+\varepsilon}^{\tau} \left[f_{q_1}\left(x_{q_1}^\varepsilon(s), u_{q_1}^o(s)\right) - f_{q_1}\left(x_{q_1}^o(s), u_{q_1}^o(s)\right)\right] ds \\ &+ \int_t^{\tau} \left[g_{q_1}\left(x_{q_1}^\varepsilon(s)\right) - g_{q_1}\left(x_{q_1}^o(s)\right)\right] dw(s). \end{aligned} \quad (22)$$

Defining the first order state variation as

$$y(\tau) := \left. \frac{d}{d\varepsilon} x^{\varepsilon}(\tau) \right|_{\varepsilon=0}, \quad (23)$$

the first order dynamics of state sensitivity are derived as

$$dy_{q_1}(\tau) = \frac{\partial f_{q_1}}{\partial x_{q_1}} \left(x^o_{q_1}(\tau), u^o_{q_1}(\tau) \right) y_{q_1}(\tau) d\tau + \frac{\partial g_{q_1}}{\partial x_{q_1}} \left(x^o_{q_1}(\tau) \right) y_{q_1}(\tau) dw(\tau), \quad (24)$$

$$y_{q_1}(t) = f_{q_1}\left(x^o_{q_1}(t), v \right) - f_{q_1}\left(x^o_{q_1}(t), u^o_{q_1}(t) \right). \quad (25)$$

Similarly, the first order (forward) cost variations are shown to be governed by

$$\frac{d}{d\tau} z_{q_1}(\tau) = \frac{\partial l_{q_1}}{\partial x_{q_1}} \left(x^o_{q_1}(\tau), u^o_{q_1}(\tau) \right) y_{q_1}(\tau) \quad (26)$$

$$z_{q_1}(t) = l_{q_1}\left(x^o_{q_1}(t), v \right) - l_{q_1}\left(x^o_{q_1}(t), u^o_{q_1}(t) \right). \quad (27)$$

It is deduced from the optimality conditions that

$$\left. \frac{d}{d\varepsilon} J(u^{\varepsilon}) \right|_{\varepsilon=0} = \mathbb{E}\left\{ z_{q_1}(t_f) + \left[\frac{\partial h}{\partial x_{q_1}} \left(x^o_{q_1}(t_f) \right) \right]^T y_{q_1}(t_f) \right\} \geq 0. \quad (28)$$

Similar to the classical case, forward and backward transition matrices (see e.g. [19]) or the Riesz Representation Theorem (see e.g. [20]), can be employed to show that there exist $\lambda^o_{q_1}, K^o_{q_1}$ such that

$$\lambda^o_{q_1}(t_f) = \frac{\partial h}{\partial x_{q_1}} \left(x^o_{q_1}(t_f) \right), \quad (29)$$

and

$$\left. \frac{d}{d\varepsilon} J(u^{\varepsilon}) \right|_{\varepsilon=0} = \mathbb{E}\left\{ z_{q_1}(t_f) + \left[\lambda^o_{q_1}(t_f) \right]^T y_{q_1}(t_f) \right\} = \mathbb{E}\left\{ z_{q_1}(t) + \left[\lambda^o_{q_1}(t) \right]^T y_{q_1}(t) \right\}, \quad (30)$$

Therefore,

$$\mathbb{E}\Big\{ l_{q_1}\left(x^o_{q_1}(t), v \right) + \left[\lambda^o_{q_1}(t) \right]^T f_{q_1}\left(x^o_{q_1}(t), v \right) - l_{q_1}\left(x^o_{q_1}(t), u^o_{q_1}(t) \right) - \left[\lambda^o_{q_1}(t) \right]^T f_{q_1}\left(x^o_{q_1}(t), u^o_{q_1}(t) \right) \Big\} \geq 0, \quad (31)$$

which results in

$$l_{q_1}\left(x^o_{q_1}(t), u^o_{q_1}(t) \right) + \left[\lambda^o_{q_1}(t) \right]^T f_{q_1}\left(x^o_{q_1}(t), u^o_{q_1}(t) \right) \leq l_{q_1}\left(x^o_{q_1}(t), v \right) + \left[\lambda^o_{q_1}(t) \right]^T f_{q_1}\left(x^o_{q_1}(t), v \right), \quad (32)$$

a.s. for all $v : \mathfrak{I}^t -$ measurable random variables in U_{q_1}. The Hamiltonian minimization condition (19) in location q_1 directly follows (32). Furthermore, the adjoint process dynamics are governed by

$$d\lambda^o_{q_1} = -\left(\frac{\partial l_{q_1}}{\partial x_{q_1}} \left(x^o_{q_1}, u^o_{q_1} \right) + \left[\frac{\partial f_{q_1}}{\partial x_{q_1}} \left(x^o_{q_1}, u^o_{q_1} \right) \right]^T \lambda^o_{q_1} + \sum_{k=1}^{n_w} \left[\frac{\partial g_{q_1}}{\partial x_{q_1}} \left(x^o_{q_1} \right) \right]^T K^{o(k)}_{q_1} \right) dt + K^o_{q_1}(t) dw(t). \quad (33)$$

Now consider a needle variation at time $t \in (t_0, t_s)$ in the form of

$$u^{\varepsilon}(\tau) = \begin{cases} u^o_{q_0}(\tau) & \text{if} \quad t_0 \leq \tau < t \\ v & \text{if} \quad t \leq \tau < t+\varepsilon \\ u^o_{q_0}(\tau) & \text{if} \quad t+\varepsilon \leq \tau < t_s - \delta^{\varepsilon} \\ u^o_{q_1}(t_s) & \text{if} \quad t_s - \delta^{\varepsilon} \leq \tau < t_s \\ u^o_{q_1}(\tau) & \text{if} \quad t_s \leq \tau \leq t_f \end{cases}. \quad (34)$$

where $\delta^{\varepsilon} \geq 0$ corresponds to the case where the perturbed trajectory arrives on the switching manifold $m(x) = 0$ at an earlier instant (the case with a later arrival time is handled in a similar fashion).

For $\tau \in [t_0, t_s - \delta^{\varepsilon})$, we may write:

$$\begin{aligned} \delta x^{\varepsilon}_{q_0}(\tau) &:= x^{\varepsilon}_{q_0}(\tau) - x^o_{q_0}(\tau) \\ &= \int_t^{t+\varepsilon} \left[f_{q_0}\left(x^{\varepsilon}_{q_0}(s), v \right) - f_{q_0}\left(x^o_{q_0}(s), u^o_{q_0}(s) \right) \right] ds \\ &+ \int_{t+\varepsilon}^{\tau} \left[f_{q_0}\left(x^{\varepsilon}_{q_0}(s), u^o_{q_0}(s) \right) - f_{q_0}\left(x^o_{q_0}(s), u^o_{q_0}(s) \right) \right] ds \\ &+ \int_t^{\tau} \left[g_{q_0}\left(x^{\varepsilon}_{q_0}(s) \right) - g_{q_0}\left(x^o_{q_0}(s) \right) \right] dw(s), \end{aligned} \quad (35)$$

and derive the first order state variation as

$$dy_{q_0}(\tau) = \frac{\partial f_{q_0}}{\partial x_{q_0}} \left(x^o_{q_0}(\tau), u^o_{q_0}(\tau) \right) y_{q_0}(\tau) d\tau \quad (36)$$

$$+ \frac{\partial g_{q_0}}{\partial x_{q_0}} \left(x^o_{q_0}(\tau) \right) y_{q_0}(\tau) dw(\tau), \quad (37)$$

$$y_{q_0}(t) = f_{q_0}\left(x^o_{q_0}(t), v \right) - f_{q_0}\left(x^o_{q_0}(t), u^o_{q_0}(t) \right). \quad (38)$$

For $\tau \in [t_s - \delta^{\varepsilon}, t_s)$, the early-switched perturbed trajectory evolves in $\mathbb{R}^{q_1}$ while the original trajectory is still in $\mathbb{R}^{q_0}$. At t_s, both trajectories are in $\mathbb{R}^{q_1}$, and we may write

$$\begin{aligned} \delta x^{\varepsilon}_{q_1}(t_s) &= x^{\varepsilon}_{q_1}(t_s) - x^o_{q_1}(t_s) \\ &= \xi\left(x^{\varepsilon}_{q_1}(t_s - \delta^{\varepsilon} -) \right) + \int_{t_s - \delta^{\varepsilon}}^{t_s} f_{q_1}\left(x^{\varepsilon}_{q_1}(\tau), u^o_{q_1}(t_s) \right) d\tau + \int_{t_s - \delta^{\varepsilon}}^{t_s} g_{q_1}\left(x^{\varepsilon}_{q_1}(\tau) \right) dw(\tau) \\ &- \xi\left(x^o_{q_1}(t_s - \delta^{\varepsilon} -) + \int_{t_s - \delta^{\varepsilon}}^{t_s} f_{q_0}\left(x^o_{q_0}(\tau), u^o_{q_0}(\tau) \right) d\tau + \int_{t_s - \delta^{\varepsilon}}^{t_s} g_{q_0}\left(x^o_{q_0}(\tau) \right) dw(\tau) \right). \end{aligned} \quad (39)$$

By invoking (3) in A1 and employing the Burkholder-Davis-Gundy (BDG) inequality (see e.g. [31], [32]) we deduce

$$y_{q_1}(t_s) = \nabla\xi \, y_{q_0}(t_s -) + \lim_{\varepsilon \to 0} \frac{\delta^{\varepsilon}}{\varepsilon} \left[f_{q_1}\left(\xi\left(x^o_{q_0}(t_s -) \right), u^o_{q_1}(t_s) \right) - \nabla\xi \, f_{q_0}\left(x^o_{q_0}(t_s -), u^o_{q_0}(t_s -) \right) \right], \quad (40)$$

almost surely, where by using (4) in A1 and the BDG inequality, the limit in (40) is determined as

$$\lim_{\varepsilon \to 0} \frac{\delta^{\varepsilon}}{\varepsilon} = \frac{\nabla m^T y_{q_0}(t_s -)}{\nabla m^T f_{q_0}\left(x^o_{q_0}(t_s -), u^o_{q_0}(t_s -) \right)}, \quad (41)$$

almost surely. Denoting

$$\gamma_s := \frac{1}{\nabla m^T f_{q_0}\left(x_{q_0}^o(t_s-), u_{q_0}^o(t_s-)\right)}, \tag{42}$$

the first order dynamics of the state sensitivity are

$$y_{q_0}(t) = f_{q_0}\left(x_{q_0}^o(t), v\right) - f_{q_0}\left(x_{q_0}^o(t), u_{q_0}^o(t)\right), \tag{43}$$

$$dy_{q_0}(\tau) = \frac{\partial f_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^o(\tau), u_{q_0}^o(\tau)\right) y_{q_0}(\tau)\, d\tau + \frac{\partial g_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^o(\tau)\right) y_{q_0}(\tau)\, dw(\tau), \tag{44}$$

$$y_{q_1}(t_s) = \left[\nabla \xi + \gamma_s\left(f_{q_1}^s - \nabla \xi\, f_{q_0}^s\right)\nabla m^T\right] y_{q_0}(t_s-), \tag{45}$$

$$dy_{q_1}(\tau) = \frac{\partial f_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^o(\tau), u_{q_1}^o(\tau)\right) y_{q_1}(\tau)\, d\tau + \frac{\partial g_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^o(\tau)\right) y_{q_1}(\tau)\, dw(\tau), \tag{46}$$

where in the above equations $f_{q_0}^s := f_{q_0}\left(x_{q_0}^o(t_s-), u_{q_0}^o(t_s-)\right)$ and $f_{q_1}^s := f_{q_1}\left(x_{q_1}^o(t_s), u_{q_1}^o(t_s)\right)$.

Furthermore, the first order dynamics of the (forward) cost sensitivity are determined by

$$z_{q_0}(t) = l_{q_0}\left(x_{q_0}^o(t), v\right) - l_{q_0}\left(x_{q_0}^o(t), u_{q_0}^o(t)\right), \tag{47}$$

$$\frac{d}{d\tau} z_{q_0}(\tau) = \frac{\partial l_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^o(\tau), u_{q_0}^o(\tau)\right) y_{q_0}(\tau), \tag{48}$$

$$z_{q_1}(t_s) = z_{q_0}(t_s-) + \left[\nabla c + \gamma_s\left(l_{q_1}^s - l_{q_0}^s - \nabla c^T f_{q_0}^s\right)\nabla m\right]^T y_{q_0}(t_s-), \tag{49}$$

$$\frac{d}{d\tau} z_{q_1}(\tau) = \frac{\partial l_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^o(\tau), u_{q_1}^o(\tau)\right) y_{q_1}(\tau). \tag{50}$$

Similar to the previous part, forward and backward transition matrices or the Riesz Representation Theorem can be employed to show that there exist $\lambda_{q_0}^o, K_{q_0}^o$ such that

$$\left.\frac{d}{d\varepsilon} J(u^\varepsilon)\right|_{\varepsilon=0} = \mathbb{E}\left\{z_{q_1}(t_f) + \left[\lambda_{q_1}^o(t_f)\right]^T y_{q_1}(t_f)\right\} = \mathbb{E}\left\{z_{q_0}(t) + \left[\lambda_{q_0}^o(t)\right]^T y_{q_0}(t)\right\}, \tag{51}$$

Therefore,

$$\mathbb{E}\Big\{l_{q_0}\left(x_{q_0}^o(t), v\right) + \left[\lambda_{q_0}^o(t)\right]^T f_{q_0}\left(x_{q_0}^o(t), v\right) - l_{q_0}\left(x_{q_0}^o(t), u_{q_0}^o(t)\right) - \left[\lambda_{q_0}^o(t)\right]^T f_{q_0}\left(x_{q_0}^o(t), u_{q_0}^o(t)\right)\Big\} \geq 0, \tag{52}$$

which results in

$$l_{q_0}\left(x_{q_0}^o(t), u_{q_0}^o(t)\right) + \left[\lambda_{q_0}^o(t)\right]^T f_{q_0}\left(x_{q_0}^o(t), u_{q_0}^o(t)\right) \leq l_{q_0}\left(x_{q_0}^o(t), v\right) + \left[\lambda_{q_0}^o(t)\right]^T f_{q_0}\left(x_{q_0}^o(t), v\right), \tag{53}$$

a.s. for all $v : \mathfrak{I}^t$ – measurable random variables in U_{q_0}. The Hamiltonian minimization condition (19) in location q_0 directly follows (53) which together with (32) completes the proof of (19) for the case under study.

The adjoint equation is given by

$$d\lambda_{q_0}^o = -\left(\frac{\partial l_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^o, u_{q_0}^o\right) + \left[\frac{\partial f_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^o, u_{q_0}^o\right)\right]^T \lambda_{q_0}^o + \sum_{k=1}^{n_w}\left[\frac{\partial g_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^o\right)\right]^T K_{q_0}^{o(k)}\right) dt + K_{q_0}^o(t)\, dw(t). \tag{54}$$

The adjoint process dynamics (14) are directly deduced from (54) and (33) together with the Hamiltonian definition (12). In order to derive the adjoint boundary conditions (18) we consider (30) for $t \downarrow t_s$ and (51) for $t \uparrow t_s$ to write

$$\mathbb{E}\left\{z_{q_1}(t_s) + \left[\lambda_{q_1}^o(t_s+)\right]^T y_{q_1}(t_s)\right\} = \mathbb{E}\left\{z_{q_0}(t_s-) + \left[\lambda_{q_0}^o(t_s)\right]^T y_{q_0}(t_s-)\right\}. \tag{55}$$

Substitution of $y_{q_1}(t_s)$ and $z_{q_1}(t_s)$ from (45) and (49) results in

$$\mathbb{E}\Big\{z_{q_0}(t_s-) + \left[\nabla c + \gamma_s\left(l_{q_1}^s - l_{q_0}^s - \nabla c^T f_{q_0}^s\right)\nabla m\right]^T y_{q_0}(t_s-) + \left[\lambda_{q_1}^o(t_s+)\right]^T \left[\nabla \xi + \gamma_s\left(f_{q_1}^s - \nabla \xi\, f_{q_0}^s\right)\nabla m^T\right] y_{q_0}(t_s-)\Big\} = \mathbb{E}\left\{z_{q_0}(t_s-) + \left[\lambda_{q_0}^o(t_s)\right]^T y_{q_0}(t_s-)\right\}. \tag{56}$$

or

$$\mathbb{E}\left\{\left[\nabla c + p\nabla m + \nabla \xi^T \lambda_{q_1}^o(t_s+) - \lambda_{q_0}^o(t_s)\right]^T y_{q_0}(t_s-)\right\} = 0, \tag{57}$$

in which the notation

$$p := \gamma_s\left(l_{q_1}^s - l_{q_0}^s - \nabla c^T f_{q_0}^s + \lambda_{q_1}^o(t_s+)^T\left(f_{q_1}^s - \nabla \xi\, f_{q_0}^s\right)\right), \tag{58}$$

is used. In order to prove the Hamiltonian continuity condition (20) we note that on one hand:

$$\begin{aligned} H_{q_0}(t_s) &\equiv H_{q_0}\left(x_{q_0}^o(t_s-), u_{q_0}^o(t_s-), \lambda_{q_0}^o(t_s), K_{q_0}^o(t_s)\right) \\ &= l_{q_0}^s + {\lambda_{q_0}^s}^T f_{q_0}^s + \mathrm{tr}\left(\left[K_{q_0}^s\right]^T g_{q_0}^s\right) \\ &= l_{q_0}^s + \left[p\nabla m + \nabla c + \nabla \xi^T \lambda_{q_1}^s\right]^T f_{q_0}^s + \mathrm{tr}\left(\left[K_{q_0}^s\right]^T g_{q_0}^s\right) \\ &= l_{q_0}^s + \gamma_s \nabla m^T f_{q_0}^s\left(l_{q_1}^s - l_{q_0}^s - \nabla c^T f_{q_0}^s + {\lambda_{q_1}^s}^T\left(f_{q_1}^s - \nabla \xi\, f_{q_0}^s\right)\right) \\ &\quad + \nabla c^T f_{q_0}^s + {\lambda_{q_1}^s}^T \nabla \xi\, f_{q_0}^s + \mathrm{tr}\left(\left[K_{q_0}^s\right]^T g_{q_0}^s\right) \\ &= l_{q_1}^s + {\lambda_{q_1}^s}^T f_{q_1}^s + \mathrm{tr}\left(\left[K_{q_0}^s\right]^T g_{q_0}^s\right), \end{aligned} \tag{59}$$

where in the derivation of the last equality γ_s is substituted from (42). On the other hand,

$$\begin{aligned} H_{q_1}(t_s) &\equiv H_{q_1}\left(x_{q_1}^o(t_s), u_{q_1}^o(t_s), \lambda_{q_1}^o(t_s+), K_{q_1}^o(t_s+)\right) \\ &= l_{q_1}^s + {\lambda_{q_1}^s}^T f_{q_1}^s + \mathrm{tr}\left(\left[K_{q_1}^s\right]^T g_{q_1}^s\right) \\ &= l_{q_1}^s + {\lambda_{q_1}^s}^T f_{q_1}^s + \mathrm{tr}\left(\left[K_{q_1}^s\right]^T \xi\left(g_{q_0}^s\right)\right) \\ &= l_{q_1}^s + {\lambda_{q_1}^s}^T f_{q_1}^s + \mathrm{tr}\left(\left[\xi\left(K_{q_1}^s\right)\right]^T g_{q_0}^s\right) \\ &= l_{q_1}^s + {\lambda_{q_1}^s}^T f_{q_1}^s + \mathrm{tr}\left(\left[K_{q_0}^s\right]^T g_{q_0}^s\right). \end{aligned} \tag{60}$$

In the derivation of the above arguments, we made use the linearity of the mapping ξ provided in A0, and we employed the assumption (3) in A1. This completes the proof of the Stochastic Hybrid Minimum Principle. ■

V. Concluding Remarks

In this paper, the Stochastic Hybrid Minimum Principle (SHMP) has been established for a general class of hybrid systems with both autonomous and controlled switchings and state jumps subject to possible changes in the dimension of the state space. The inevitability of switchings and jumps upon arrival on switching manifolds is of particular importance in the modelling of mechanical impact problems (e.g. [29] as well as the celebrated bouncing ball example) and friction-resisted dynamical systems with distinct evolutions under static and dynamic frictions (see e.g. [14]). The SHMP established here generalizes the deterministic HMP presented in [12]–[15], [27]–[29]. Furthermore, as proved in the case of deterministic hybrid optimal control problems (see e.g. [13], [15]), the adjoint process in the HMP and the gradient of the value function in Hybrid Dynamic Programming (HDP) are indentical to each other almost everywhere. So due to the fact that the same relationship holds for continuous parameter stochastic optimal control problems (see e.g. [21]), it is natural to expect the adjoint process in the SHMP and the gradient of the value function in Stochastic HDP (SHDP) to be identical almost everywhere. Indeed, the formulation of SHDP and the investigation of its relationship to the SHMP is the subject of another study expected to be presented in a consecutive paper.

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