



A class of hybrid LQG mean field games with state-invariant switching and stopping strategies[☆]

Dena Firoozi^{a,*}, Ali Pakniyat^b, Peter E. Caines^c

^a Department of Decision Sciences, HEC Montréal, Montreal, QC, Canada

^b Department of Mechanical Engineering, University of Alabama, Tuscaloosa, AL, USA

^c Centre for Intelligent Machines (CIM) and the Department of Electrical and Computer Engineering (ECE), McGill University, Montreal, QC, Canada

ARTICLE INFO

Article history:

Received 11 October 2020
Received in revised form 5 September 2021
Accepted 7 January 2022
Available online 7 April 2022

Keywords:

Mean field games
Hybrid optimal control
Switching and stopping times

ABSTRACT

A novel framework is presented that combines Mean Field Game (MFG) theory and Hybrid Optimal Control (HOC) theory to obtain a unique ε -Nash equilibrium for a non-cooperative game with switching and stopping times. We consider the case where there exists one major agent with a significant influence on the system together with a large number of minor agents constituting two subpopulations, each agent with individually asymptotically negligible effect on the whole system. Each agent has stochastic linear dynamics with quadratic costs, and the agents are coupled in their dynamics and costs by the average state of minor agents (i.e. the empirical mean field). It is shown that for a class of Hybrid LQG MFGs, the optimal switching and stopping times are state-invariant and only depend on the dynamical parameters of each agent. Accordingly, a hybrid systems formulation of the game is presented via the indexing by discrete events: (i) the switching of the major agent between alternative dynamics or (ii) the termination of the agents' trajectories in one or both of the subpopulations of minor agents. Optimal switchings and stopping time strategies together with best response control actions for, respectively, the major agent and all minor agents are established with respect to their individual cost criteria by an application of Hybrid LQG MFG theory.

© 2022 Elsevier Ltd. All rights reserved.

1. Introduction

Mean Field Game (MFG) theory studies the existence of approximate Nash equilibria and the corresponding individual strategies for stochastic dynamical systems in games involving a large number of agents. Basically, the theory exploits the relationship between the large finite and the corresponding infinite limit population problems. The equilibria are termed ε -Nash equilibria and are generated by the local, limited information control actions of each agent in the population. The control actions constitute the best response of each agent with respect to

the behaviour of the mass of agents. Moreover, the approximation error, induced by using the MFG solution, converges to zero as the population size tends to infinity.

The analysis of this set of problems originated in Huang, Caines, and Malhamé (2003, 2007) and Huang, Malhamé, and Caines (2006), and independently in Lasry and Lions (2006a, 2006b, 2007). Many extensions and generalizations of MFGs exist, principally the probabilistic formulation (Carmona & Delarue, 2018), the master equation approach (Cardaliaguet, Delarue, Lasry, & Lions, 2019) and mean field type control theory (Bensoussan, Frehse, & Yam, 2013). In Huang (2010) and Nguyen and Huang (2012) the authors analyse and solve the completely observed (CO) linear quadratic Gaussian (LQG) systems case where there is a major agent (i.e. non-asymptotically vanishing as the population size goes to infinity) together with a population of minor agents (i.e. individually asymptotically negligible). The existence of closed-loop ε -Nash equilibria is established together with the individual agents' control laws that yield the equilibria (Nguyen & Huang, 2012). A convex analysis method is utilized in Firoozi, Jaimungal, and Caines (2020) to retrieve the solutions of Huang (2010), where no assumption is imposed on the evolution of the mean field a priori. The CO MM nonlinear (NL) MFG problem is treated in Nourian and Caines (2013). This framework is further extended in Caines and

[☆] This work was supported in part by the Fonds de Recherche du Québec-Nature et Technologies (funding reference number 258061), Canada, the Natural Sciences and Engineering Research Council (funding reference number 2019-05336), Canada, and the Air Force Office of Scientific Research (funding reference number FA9550-19-1-0138), USA. The material in this paper was partially presented at the 56th IEEE Conference on Decision and Control, December 12–15, 2017, Melbourne, Australia, the 20th IFAC World Congress of the International Federation of Automatic Control, July 9–14, 2017, Toulouse, France. This paper was recommended for publication in revised form by Associate Editor Alessandro Abate under the direction of Editor Ian R. Petersen.

* Corresponding author.

E-mail addresses: dena.firoozi@hec.ca (D. Firoozi), apakniyat@ua.edu (A. Pakniyat), peterc@cim.mcgill.ca (P.E. Caines).

Kizilkale (2017), Firoozi (2019), Firoozi and Caines (2017, 2019, 2021) and Şen and Caines (2016) for partially observed MFG theory for nonlinear and linear quadratic systems. Using the probabilistic approach to MFGs, Carmona and Wang (2017) and Carmona and Zhu (2016) establish the existence of open-loop and closed-loop ε -Nash equilibria for a general MM MFG and provide explicit solutions for an LQG case. The works (Cardaliaguet, Cirant, & Porretta, 2018; Lasry & Lions, 2018) characterize the Nash equilibrium for a general MFG system with one major agent and an infinite number of minor agents via the MFG Master Equations. It is to be noted that for the LQG case it has been, respectively, demonstrated in Firoozi (2020) and Huang (2020) that the (Markovian) closed-loop solutions to LQG MM MFGs obtained through the master equation and the probabilistic approaches are identical to the original LQG MM MFG solutions of Huang (2010). (Another line of research characterizes a Stackelberg equilibrium between the major agent and the minor agents, see e.g. Bensoussan, Chau, Lai, and Yam (2017) and Moon and Başar (2018).)

MFGs have found numerous applications in particular in mathematical finance and economics for characterizing equilibrium price and market equilibria (see Casgrain and Jaimungal (2020), Firoozi and Caines (2017), Gomes, Lafleche, and Nurbekyan (2016), Shrivats, Firoozi, and Jaimungal (2022a, 2022b) and Carmona (2020) and the references therein) – to name a few.

In several situations in stochastic dynamic games, such as in financial markets (Firoozi, Pakniyat, & Caines, 2017), agents wish to find the best time at which to enter or exit a given strategy. In order to determine the optimal stopping time strategies together with best response policies for the agents one is required to invoke the necessary optimality conditions of stochastic hybrid optimal control theory (Aghayeva & Abushov, 2011; Bensoussan & Menaldi, 2000; Pakniyat & Caines, 2016, 2017a). These optimality conditions are an extension of deterministic hybrid optimal control theory (Bensoussan & Menaldi, 1997; Garavella & Piccoli, 2005; Pakniyat & Caines, 2017c, 2021; Shaikh & Caines, 2007; Sussmann, 1999; Taringoo & Caines, 2013) for systems interacting with stochastic diffusions. In Pakniyat and Caines (2016), in particular, the Stochastic Hybrid Minimum Principle (SHMP) is established for a general class of stochastic hybrid systems with both autonomous and controlled switchings and jumps possibly accompanied by dimension changes. Given the computational difficulty in solving general nonlinear forward–backward stochastic differential equations (FB-SDE) and the associated boundary conditions via the SHMP, a class of linear quadratic Gaussian (LQG) HOC problems is presented in Pakniyat and Caines (2017a) for which the corresponding Riccati equations are independent of the realizations of the stochastic diffusion terms.

The first combination of Mean Field Game (MFG) theory and Hybrid Optimal Control (HOC) theory appeared in the predecessor to the current paper, Firoozi et al. (2017); in that analysis a non-cooperative game formulation of electronic markets was presented, where high frequency traders (HFTs) may leave the market before a given final time. The best stopping time policies for the traders are further shown to yield a closed-loop ε -Nash equilibrium for the market. The advances in this paper beyond the contributions in Firoozi et al. (2017) are as follows:

- The system considered is a general time-varying LQG mean field game system with switching and stopping time strategies. As such, the time derivative of the switching cost weight matrix appears in the switching (stopping) Eq. (11) (Eq. (15)).
- Both sufficient and necessary optimality conditions for a switching (stopping) to take place are provided in Theorem 1 and Corollary 2.

- The proofs of Theorem 1, Corollary 2, and Theorem 3 (ε -Nash property) are presented.
- A numerical methodology is developed for solving the set of hybrid MFG equations in Section 4.5 and is implemented for an example in Section 5.
- The major agent is provided with the option to switch to different dynamics which leads to more complex automata. More specifically, the system has 4 discrete states with an associated increase in the number of potential realizations compared to the case study in Firoozi et al. (2017) where there exist 3 discrete states.

We observe that for general Hybrid LQG problems (including LQG stopping systems as the special case with switching to zero dynamics), the optimal switching (and stopping) times are filtration-adapted random variables and, hence, optimal inputs are not necessarily representable in a Riccati format. For such problems, nonlinear versions of hybrid mean field games may be formulated and solved, which is beyond the scope of the current paper and is the subject of future work. However, as discussed in Theorem 1 (and Corollary 2), for certain classes of hybrid (and stopping) LQG systems, the optimality conditions of the SHMP yield state-invariant representations of the optimal switching (and stopping) times, which can be identified deterministically based upon the dynamical parameters of each agent. Hence, in the limiting MFG formulation of the problem all minor agents within the same subpopulation stop at the same time yielding a deterministic representation of the mean field. Subsequently, a hybrid formulation of the game is developed for which switching events correspond to (i) the switching of the major agent or (ii) the cessation of one or both subpopulations of minor agents. Hence, by developing and then utilizing a hybrid LQG MFG theory, optimal switching and stopping time strategies for, respectively, the major agent and all minor agents, together with their best response control actions which yield a unique ε -Nash equilibrium are established.

A recent work (Bouveret, Dumitrescu, & Tankov, 2020) studies the stopping of agents in the infinite-population MFG systems, where a relaxed solution approach is followed by looking for the occupation measure of agents instead of their stopping time. Kordonis and Papavassilopoulos (2015) study MFG systems where at each time instant a random number of agents enter and remain in the system for a specific time duration. The random entrance is described by a Markov chain and the problem is formulated as an LQG optimal control problem for Markov jump linear systems.

We note that the following terms are used interchangeably throughout the paper: optimal and best response in the infinite-population case, quit and stop, control action and control input.

The paper organization is as follows: Section 2 introduces single-agent hybrid LQG systems with state-invariant switching and stopping strategies. Subsequently, Section 3 presents the class of hybrid LQG MFG problems under study. More specifically, this section is devoted to the two types of transitions that exist between the dynamics at the individual level, i.e. at a transition event the major agent switches dynamics (from one of the realizations of Eq. (22) to the other) or a minor agent stops (switches from dynamics (26) to zero dynamics). Section 4 presents hybrid-MFG approach, where, at each discrete state, the major agent's state is extended by the corresponding mean field, and a generic minor agent's state is extended by the corresponding major agent's state and the mean field. At the mean field game level the dynamics governing the extended states are undergoing changes, in which case, the associated dynamics are presented in Sections 4.2.1 and 4.3.1, the corresponding transitions in Section 4.2.2 and Appendix B, and the best-response solutions in Sections 4.2.3 and 4.3.2, respectively, for the major agent and a

generic minor agent. Subsequently, the hybrid-MFG consistency equation and ε -Nash property are presented in Section 4.4. Next, Section 5 depicts simulation results. Finally, Section 6 presents concluding remarks and future directions.

2. State-invariant optimal switching and stopping strategies for single-agent hybrid LQG systems

In this section single-agent hybrid LQG systems are presented. Then, a set of sufficient conditions (stemming from Pakniyat & Caines, 2017a) are derived under which the optimal switching and stopping times for such systems are state-invariant. The results are subsequently used in the formulation of a class of hybrid LQG MFGs in the rest of the paper.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ be a probability space such that \mathcal{F}_0 contains the P -null sets, $\mathcal{F}_T = \mathcal{F}$ for a fixed final time $T < \infty$, and let \mathcal{F}_t be the natural filtration associated with the sigma-algebra generated by a Wiener process up to time t .

The (hybrid) state of a stochastic hybrid system is denoted by $h = (Q, x)$ where $Q \in \mathbb{Q}$ denotes the discrete state (component) taking values from \mathbb{Q} with finite cardinality, and $x \in \mathbb{R}^{n_Q}$ denotes the continuous component of the hybrid state (shortly referred to as the continuous state). We introduce the *counting index* $j \in \mathbb{Z}_{\geq 0}$ that indicates the number of switchings incurred within the interval $[t_0, t)$. Conversely, denoting by t_j the j th switching instant, the expression $t \in [t_j, t_{j+1})$ indicates that the value of Q has changed (switched) j times by time t . In this paper, all changes in the value of Q are controlled switchings, i.e. every switching is a direct consequence of a control action.

A *hybrid input process* is a pair $(S_L, u(\cdot)) =: I_L \equiv I_L^{[0, T]}$ defined on $[0, T]$, $T < \infty$, where $S_L = ((t_0, Q_0), (t_1, Q_1), \dots, (t_L, Q_L))$, $L < \infty$, is a finite *hybrid sequence of switching events* consisting of a strictly increasing sequence of \mathcal{F}_t -adapted times t_j , and $u(\cdot) := \{u^{Q_0}(\cdot), u^{Q_1}(\cdot), \dots, u^{Q_L}(\cdot)\}$ is an \mathcal{F}_t -adapted continuous input process, where for every $t \in [t_j, t_{j+1})$, $j \in \{0, 1, \dots, L\}$ (equivalently denoted by $0 \leq j \leq L$ or $j = 0, 1, \dots, L$), the continuous input $u^{Q_j}(t)$ is an \mathcal{F}_t -adapted, $\mathbb{R}^{m_{Q_j}}$ valued, random variable.

The dynamics of the continuous state process are governed by linear Itô differential equations of the form

$$dx^{Q_j}(t) = (A^{Q_j}(t)x^{Q_j}(t) + B^{Q_j}(t)u^{Q_j}(t))dt + D^{Q_j}(t)dw(t), \quad t \in [t_j, t_{j+1}), \quad (1)$$

where $Q_j \in \mathbb{Q}$, $x^{Q_j}(t) \in \mathbb{R}^{n_{Q_j}}$, $u^{Q_j}(t) \in \mathbb{R}^{m_{Q_j}}$, $w(t) \in \mathbb{R}$, $A^{Q_j}(t) \in \mathbb{R}^{n_{Q_j} \times n_{Q_j}}$, $B^{Q_j}(t) \in \mathbb{R}^{n_{Q_j} \times m_{Q_j}}$, $D^{Q_j}(t) \in \mathbb{R}^{n_{Q_j}}$, $0 \leq j \leq L$, $t_0 := 0$, $t_{L+1} := T$. Upon switching at a switching time t_j , the continuous component of the state is reinitialized according to a jump map provided as

$$x^{Q_j}(t_j) = \Psi_{\sigma_j} x^{Q_{j-1}}(t_j^-) \equiv \Psi_{\sigma_{Q_{j-1}, Q_j}} x^{Q_{j-1}}(t_j^-). \quad (2)$$

It is further assumed that

$$D^{Q_j}(t_j) = \Psi_{\sigma_{Q_{j-1}, Q_j}} D^{Q_{j-1}}(t_j), \quad (3)$$

for all $1 \leq j \leq L$, which implies equivalent diffusion fields before and after switching events.

Given an initial condition $(Q(0), x^{Q_0}(0)) = (Q_0, x_0^{Q_0})$, the cost associated with the hybrid input I_L over the time horizon $[0, T]$ is considered to be of the form

$$J(t_0, (Q_0, x_0^{Q_0}), I_L) = \frac{1}{2} \mathbb{E} \left[\|x^{Q_L}(T)\|_{\bar{P}^{Q_L}(T)}^2 + \sum_{j=1}^L \|x^{Q_{j-1}}(t_j^-)\|_{C_{\sigma_j}^{Q_j}(t_j)}^2 \right]$$

$$+ \sum_{i=0}^L \int_{t_i}^{t_{i+1}} \left(\|x^{Q_i}(t)\|_{P^{Q_i}(t)}^2 + \|u^{Q_i}(t)\|_{R^{Q_i}(t)}^2 \right) dt, \quad (4)$$

where $0 \leq [\bar{P}^{Q_L}(t)]^T = \bar{P}^{Q_L}(t) \in \mathbb{R}^{n_{Q_L} \times n_{Q_L}}$, $0 \leq [C_{\sigma_j}(t)]^T = C_{\sigma_j}(t) \in \mathbb{R}^{n_{Q_{j-1}} \times n_{Q_{j-1}}}$, $0 \leq [P^{Q_i}(t)]^T = P^{Q_i}(t) \in \mathbb{R}^{n_{Q_i} \times n_{Q_i}}$, $0 < [R^{Q_i}(t)]^T = R^{Q_i}(t) \in \mathbb{R}^{m_{Q_i} \times m_{Q_i}}$. The associated stochastic hybrid optimal control problem is to find $\inf_{I_L} J(t_0, (Q_0, x_0^{Q_0}), I_L)$.

Theorem 1 (Sufficient Conditions for \mathcal{F}_t -invariance of Optimal Solutions of the Hybrid LQG Problem). For the system governed by (1)–(4), assume that a family of matrices $\{\Pi^{Q_j}(t); j = 0, 1, \dots, L\}$ exists satisfying the following family of Riccati equations (for simplicity of notation, the explicit time dependence (t) is dropped whenever it is clear from the context)

$$\dot{\Pi}^{Q_j} = \Pi^{Q_j} B^{Q_j} [R^{Q_j}]^{-1} [B^{Q_j}]^T \Pi^{Q_j} - \Pi^{Q_j} A^{Q_j} - [A^{Q_j}]^T \Pi^{Q_j} - P^{Q_j}, \quad (5)$$

subject to the terminal and boundary conditions

$$\Pi^{Q_L}(T) = \bar{P}^{Q_L}, \quad (6)$$

$$\Pi^{Q_{j-1}}(t_j) = \Psi_{\sigma_j}^T \Pi^{Q_j}(t_j) \Psi_{\sigma_j} + C_{\sigma_j}(t_j), \quad (7)$$

and for every $j = L, L-1, \dots, 1$, there exist $t_j \in [0, t_{j+1})$ satisfying the following algebraic matrix relations (equality, strict positive definiteness, and strict negative definiteness):

$$H_{\sigma_j}^{\Delta}(s) = 0, \quad s = t_j, \quad (8)$$

$$H_{\sigma_j}^{\Delta}(s) > 0, \quad s > t_j, \quad (9)$$

$$H_{\sigma_j}^{\Delta}(s) < 0, \quad s < t_j, \quad (10)$$

where the time order of the strict matrix inequalities corresponds to the strict decrease in the value function, and

$$H_{\sigma_j}^{\Delta}(s) := \Psi_{\sigma_j}^T \Pi^{Q_j}(s) \left[B^{Q_j} [R^{Q_j}]^{-1} [B^{Q_j}]^T - \Psi_{\sigma_j} B^{Q_{j-1}} [R^{Q_{j-1}}]^{-1} [B^{Q_{j-1}}]^T \Psi_{\sigma_j}^T \right] \Pi^{Q_j}(s) \Psi_{\sigma_j} + \Psi_{\sigma_j}^T \Pi^{Q_j}(s) \left[\Psi_{\sigma_j} A^{Q_{j-1}} - A^{Q_j} \Psi_{\sigma_j} - \Psi_{\sigma_j} B^{Q_{j-1}} [R^{Q_{j-1}}]^{-1} [B^{Q_{j-1}}]^T C_{\sigma_j} \right] + \left[[A^{Q_{j-1}}]^T \Psi_{\sigma_j} - \Psi_{\sigma_j}^T [A^{Q_j}]^T - C_{\sigma_j} B^{Q_{j-1}} [R^{Q_{j-1}}]^{-1} [B^{Q_{j-1}}]^T \Psi_{\sigma_j}^T \right] \Pi^{Q_j}(s) \Psi_{\sigma_j} + P^{Q_{j-1}} - C_{\sigma_j} B^{Q_{j-1}} [R^{Q_{j-1}}]^{-1} [B^{Q_{j-1}}]^T C_{\sigma_j} + C_{\sigma_j} A^{Q_{j-1}} + [A^{Q_{j-1}}]^T C_{\sigma_j} - \Psi_{\sigma_j}^T P^{Q_j} \Psi_{\sigma_j} - \frac{\partial C_{\sigma_j}(t)}{\partial t} \Big|_{t=s}. \quad (11)$$

Then the switching times are \mathcal{F}_t -independent (almost surely deterministic) and are independent of the initial conditions, and the associated optimal control actions are determined as

$$u^{Q_j, \circ}(t, x) = -[R^{Q_j}(t)]^{-1} [B^{Q_j}(t)]^T \Pi^{Q_j}(t) x^{Q_j, \circ}(t). \quad \square \quad (12)$$

Proof. See Appendix A.

An important consequence of Theorem 1 is that it yields as a corollary the crucial existence condition for the optimal stopping times used in Theorem 3.

Consider a system governed by

$$dx(t) = (A(t)x(t) + B(t)u(t))dt + D(t)dw(t), \quad (13)$$

where $t \in [0, t_s^\omega)$, and t_s^ω is an \mathcal{F}_t -adapted stopping time, to be determined together with a continuous input in order to minimize (minimize) the cost

$$J(u) = \frac{1}{2} \mathbb{E} \left[\|x(t_s^\omega)\|_{C(t_s^\omega)}^2 + \int_0^{t_s^\omega} (\|x(t)\|_{P(t)}^2 + \|u(t)\|_{R(t)}^2) dt \right], \quad (14)$$

Define

$$H^\Delta(s) := P(s) - C(s)B(s)R^{-1}(s)B^T(s)C(s) + C(s)A(s) + A^T(s)C(s) - \left. \frac{\partial C(t)}{\partial t} \right|_{t=s}. \quad (15)$$

Corollary 2 (Stopping Policies for LQG Systems). Consider the (deterministic) algebraic matrix expression (15). If there exists a finite time $t_s \in [0, \infty)$ for which

$$H^\Delta(s) = 0, \quad s = t_s, \quad (16)$$

$$H^\Delta(s) > 0, \quad s > t_s, \quad (17)$$

$$H^\Delta(s) < 0, \quad s < t_s, \quad (18)$$

then $t_s^\omega = t_s$ for almost all $\omega \in \Omega$, that is to say, the optimal stopping time for the system (13) with the cost (14) is \mathcal{F}_t -independent, state-invariant, and takes the value t_s almost surely, and the optimal input is determined by

$$u(t, x) = -R^{-1}(t)B^T(t)\Pi(t)x(t), \quad (19)$$

where $\Pi(t)$ is the solution to

$$\dot{\Pi} = \Pi B R^{-1} B^T \Pi - \Pi A - A^T \Pi - P, \quad (20)$$

subject to the terminal (stopping) condition

$$\Pi(t_s) = C(t_s). \quad \square \quad (21)$$

Proof. The proof is immediate since it expresses the conditions of Theorem 1 for the special case of $\Psi_\sigma = 0$.

3. Major-minor hybrid LQG mean field games

3.1. Problem description

We consider the case where there exists one major agent and N minor agents interacting with each other through the mean field coupling in their dynamics over the time interval $[0, T]$. Two types of minor agents are considered: type \mathcal{A}^a with the population of N_a and type \mathcal{A}^b with the population of N_b , such that $N_a + N_b = N$.

The dynamics of the major agent and a generic minor agent are described by the linear time evolution of their states and a quadratic performance function. However, the two populations of minor agents have different linear dynamics and quadratic performance objectives. We study the case where the major agent \mathcal{A}_0 is permitted to switch from one set of dynamics to another at time t_s^0 if optimal, while a generic minor agent \mathcal{A}_i , $1 \leq i \leq N$, is permitted to stop at an optimal time t_s^i . With abuse of notation, the superscript k in \mathcal{A}_0^k , $k = 1, 2$, denotes that the major agent's operation mode governed by the dynamics (22) and the cost functional (24), and in \mathcal{A}_i^k , $1 \leq i \leq N$, $k \triangleq a, b$, denotes that minor agent \mathcal{A}_i , $1 \leq i \leq N$, is of type k , $k \triangleq a, b$, governed by (26)–(27). As discussed in Section 3.2, the optimal switching or stopping time policy for each agent is trajectory and state independent, and depends only on its dynamical parameters (i.e. the agent's type). Since the dynamical parameters for all minor agents in their respective types are the same, it follows that the stopping times are the same for all agents of each subpopulation. The distinct nature of the switching (stopping) events, together with the continuous evolution of the state processes between switchings,

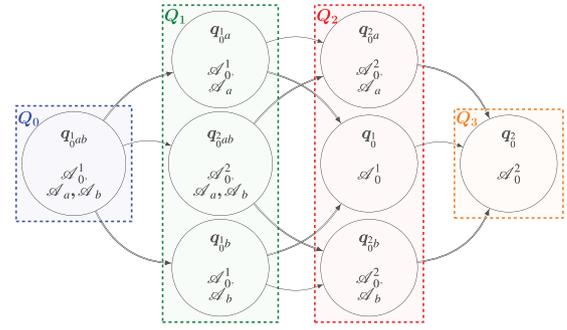


Fig. 1. Hybrid Automata Diagram with a single major player and two populations of minor players with switching and stopping times. Transitions accompanied by dimension changes are identified with double-line arrows.

result in the stochastic hybrid form of the problem analysed in this paper. Moreover, the fact that the minor agents are modelled as members of large populations gives rise to our use of the LQG MFG framework. The system has several distinct combinatoric alternatives; this is because there are various distinct sequences wherein one subpopulation of minor agents or another drops out first, or the major agent switches to one particular discrete state before or after a minor agent stopping event. It is to be emphasized that the discrete state sequence that actually occurs for any given system depends upon the solution of the complete (initial to terminal) hybrid MFG equations for the system, and in particular is not prescribed. We note that a key condition which yields the collective switching of the entire subpopulations is given by (3) (see Section 2) and while this is reasonable in a class of LQG problems, the corresponding condition is most unlikely to hold in a nonlinear framework.

3.2. Discrete state association

In order to present the dynamics of the system in the stochastic hybrid systems framework of Pakniyat and Caines (2016, 2017a), the discrete states $q_{k,\bullet}$ are assigned (see Fig. 1) where $k \triangleq a, b$ refers to the mode in the dynamics of the major agent and \bullet represents the active subpopulations of minor agents. For instance, the discrete state $q_{1,ab}$ indicates that the major agent is subject to its first dynamics and both subpopulations \mathcal{A}^a and \mathcal{A}^b are present, and the discrete state $q_{2,a}$ indicates that the major agent is subject to its second dynamics, subpopulation \mathcal{A}^a is present and subpopulation \mathcal{A}^b has already quit the system. Furthermore, in order to refer to the temporal mode of the system, the multivalued discrete states Q_j , $0 \leq j \leq 3$, are introduced (see Fig. 1), which correspond to the evolution of the system within the intervals $[t_j, t_{j+1})$, where $t_0 = 0$ is the initial time, t_1, t_2, t_3 correspond to the times of the events of stopping of a subpopulation or switching of the major agent, in the order of occurrence, and $t_4 = T$ is the terminal time. This corresponds to the scenario in which all the possible discrete changes in the system occur before the terminal time, i.e. $Q_3 = q_0^2$. Other scenarios where the discrete state at terminal time is different from the case considered here are possible with minor variations over the results presented in this paper.

Now, we describe the evolution of the system over the sequence of generic discrete states Q_j , $0 \leq j \leq 3$. The discrete state Q_0 , as indicated in Fig. 1, associates with the system evolution over the interval $[0, t_1)$ in the system's initial setting where both subpopulations of minor agents are interacting together and with the major agent which is subject to its first dynamics \mathcal{A}_0^1 .

The multivalued discrete state Q_1 corresponds to the evolution of the system over $[t_1, t_2)$ with one change relative to the initial setting; this consists of three possible situations: (i) the major agent subject to its second dynamics \mathcal{A}_0^2 is interacting with both subpopulations $\mathcal{A}^a, \mathcal{A}^b$ present in the system; this corresponds to the centre node inside Q_1 in Fig. 1 and is denoted by $Q_1 = q_{2_{0ab}}$, (ii) the major agent subject to its first dynamics \mathcal{A}_0^1 is interacting with the subpopulation \mathcal{A}^a while the subpopulation \mathcal{A}^b has quit the system; this corresponds to the top node inside Q_1 in Fig. 1 and is denoted by $Q_1 = q_{1_{0a}}$, and (iii) the major agent subject to its first dynamics \mathcal{A}_0^1 is interacting with \mathcal{A}^b while \mathcal{A}^a has quit, corresponding to the bottom node inside Q_1 in Fig. 1, denoted by $Q_1 = q_{1_{0b}}$.

The multivalued discrete state Q_2 represents the evolution of the system over $[t_2, t_3)$ with two changes relative to the initial setting for which three situations can be considered: (I) the major agent subject to its second dynamics \mathcal{A}_0^2 is interacting with the subpopulation \mathcal{A}^a , and the subpopulation \mathcal{A}^b has already quit, which corresponds to the top node inside Q_2 in Fig. 1 denoted as $Q_2 = q_{2_{0a}}$, (II) the major agent subject to its second dynamics \mathcal{A}_0^2 is interacting with \mathcal{A}^b , and the subpopulation \mathcal{A}^a has already quit, which corresponds to the bottom node inside Q_2 in Fig. 1 denoted by $Q_2 = q_{2_{0b}}$, (III) the major agent is subject to its first dynamics \mathcal{A}_0^1 and both subpopulations $\mathcal{A}^a, \mathcal{A}^b$ have already quit, which corresponds to the centre node inside Q_2 in Fig. 1, denoted by $Q_2 = q_0$.

The discrete state Q_3 corresponds to the evolution of the major agent subject to its second dynamics \mathcal{A}_0^2 over $[t_3, T]$ which corresponds to $Q_3 = q_2$.

In this work it is assumed that each of the time periods $[t_j, t_{j+1})$ associated with the multivalued discrete state $Q_j, 0 \leq j \leq 3$, is non-empty, i.e. $t_j < t_{j+1}$. This assumption is tenable since it will be shown that for the class of hybrid LQG systems in this paper, the switching times t_1, t_2, t_3 can be deterministically evaluated as they depend only on the system parameters.

Remark 1. It would be possible to extend the formulation to include a larger number of subpopulations greater than two. However, this results in a larger number of discrete states and associated realizations for the system. Therefore the corresponding automata becomes more complex and the computational load increases. The problem may become intractable if the number of subpopulations becomes large.

3.3. Dynamics and costs: Finite population

3.3.1. Major agent

Let the evolution of the major agent $\mathcal{A}_0^k, k = 1, 2$, be expressed as

$$dx_0 = A_0^k x_0 dt + B_0^k u_0 dt + F_0^k x^{(N_t)} dt + D_0^k dw_0, \quad (22)$$

where $x_0 \in \mathbb{R}^n$ is the state, $u_0 \in \mathbb{R}^m$ is the control input, and $w_0 \in \mathbb{R}^r$ is a standard Wiener process. The matrices A_0^k, B_0^k, F_0^k , and $D_0^k, k = 1, 2$, are of appropriate dimension.

From (22), the major agent is coupled with the minor agents by the average term $x^{(N_t)} = \frac{1}{N_t} \sum_{i=1}^{N_t} x_i$. Note that in (22), N_t may take the following values.

$$N_t = \begin{cases} N_a + N_b & \text{for } Q_0 = q_{1_{0ab}}, Q_1 = q_{2_{0ab}} \\ N_a & \text{for } Q_1 = q_{1_{0a}}, Q_2 = q_{2_{0a}} \\ N_b & \text{for } Q_1 = q_{1_{0b}}, Q_2 = q_{2_{0b}} \\ 0 & \text{for } Q_2 = q_1, Q_3 = q_2, \end{cases} \quad (23)$$

where $x^{(N_t)} := 0$ for $N_t = 0$. The major agent $\mathcal{A}_0^k, k = 1, 2$, aims to minimize the following cost functional

$$J_0^{N_t, k}(u_0, u_{-0}) = \frac{1}{2} \mathbb{E} \left[\|x_0(T)\|_{\bar{P}_0^k}^2 + \int_0^T (\|x_0 - \Phi(x^{(N_t)})\|_{P_0^k}^2 + \|u_0\|_{R_0^k}^2) dt \right], \quad (24)$$

$$\Phi(x^{(N_t)}) := H_0^k x^{(N_t)}, \quad (25)$$

with the weight matrices of appropriate dimension.

Assumption 1. (i) $(P_0^k)^\top = P_0^k, (R_0^k)^\top = R_0^k, (\bar{P}_0^k)^\top = \bar{P}_0^k$, (ii) the matrices $A_0^k, B_0^k, F_0^k, P_0^k, H_0^k$ and R_0^k are bounded, (iii) R_0^k is a continuous function of time $t, 0 \leq t \leq T$, and (iv) every column $D_0^k(:, j), j = 1, \dots, r$, of D_0^k is such that $\int_0^T \|D_0^k(:, j)\|^2 < \infty$.

Assumption 2 (Convexity). $R_0^k > 0, \bar{P}_0^k \geq 0$, and $P_0^k \geq 0$.

We note once again that the superscript k in \mathcal{A}_0^k denotes that the major agent is acting with respect to the dynamics and the cost functional k .

3.3.2. Generic \mathcal{A}^k -type minor agent

The dynamics for a minor agent $\mathcal{A}_i^k, k \triangleq a, b$, is given by

$$dx_i = A_k x_i dt + B_k u_i dt + G_k x_0 dt + F_k x^{(N_t)} dt + D_k dw_i, \quad (26)$$

where $x_i \in \mathbb{R}^n$ is the state of agent $\mathcal{A}_i^k, u_i \in \mathbb{R}^m$ is the control input, $w_i \in \mathbb{R}^r$ is a standard Wiener process, and A_k, B_k, G_k, F_k, D_k are constant matrices of appropriate dimension. Note that N_t in (26) again takes values as in (23) over the horizon $[0, T]$. The cost for a type \mathcal{A}^k minor agent is given by

$$J_i^{N_t, k}(u_i, u_{-i}) = \frac{1}{2} \mathbb{E} \left[\|x_i(t_s^i) - \Psi_k(x^{(N_t)})\|_{\bar{P}_k}^2 + \int_0^{t_s^i} (\|x_i - \Psi_k(x^{(N_t)}, x_0)\|_{P_k}^2 + \|u_i\|_{R_k}^2) dt \right], \quad (27)$$

$$\Psi_k(x^{(N_t)}, x_0) := H_1^k x_0(t) + H_2^k x^{(N_t)}, \quad (28)$$

where the weight matrices have appropriate dimensions, and $t_s^i \in [0, T]$ is the stopping time of agent i freely decided by this agent in order to minimize its individual cost. Particularly, t_s^i is not directly restricted to be the same over the entire population. However, whenever the parameters of the associated extended dynamics (presented in Section 4.3) satisfy the requirements of Corollary 2, then the optimal stopping times $t_s^i = t_s^k$ for the entire subpopulation of \mathcal{A}^k -type minor agents.

Assumption 3. (i) $P_k^\top = P_k, R_k^\top = R_k, \bar{P}_k^\top = \bar{P}_k$, (ii) the matrices $A_k, B_k, F_k, G_k, P_k, \bar{P}_k, H_1^k, H_2^k$ and R_k are bounded, (iii) R_k is a continuous function of time $t, 0 \leq t \leq T$, and (iv) every column $D_k(:, j), j = 1, \dots, r$, of D_k is such that $\int_0^T \|D_k(:, j)\|^2 < \infty$.

Assumption 4 (Convexity). $\bar{P}_k \geq 0, P_k \geq 0$, and $R_k > 0$.

From (26) and (27), a generic \mathcal{A}^k -type minor agent interacts with the major agent's state as well as the average state of all existing minor agents through its dynamics and cost functional.

We denote by $w = \{w_i, 0 \leq i \leq N\}$ the set of $(N + 1)$ independent \mathbb{R}^r -valued standard Wiener processes on the probability space (Ω, \mathcal{F}, P) , where w is progressively measurable with respect to the filtration $\mathcal{F}^w = \{\mathcal{F}_t^w \subset \mathcal{F}; t \geq 0\}$.

Assumption 5. The initial states $\{x_i(0), 0 \leq i \leq N\}$ defined on (Ω, \mathcal{F}, P) are identically distributed, mutually independent and also independent of \mathcal{F}_T^w , with $\mathbb{E}x_i(0) = 0$. Moreover, $\sup_i \mathbb{E}\|x_i(0)\|^2 \leq c < \infty, 0 \leq i \leq N < \infty$, with c independent of N .

We also define

$$\mathcal{S}_k := \{1 \leq i \leq N \mid \text{agent } i \text{ belongs to sub-population } k\},$$

where $N_k := |\mathcal{S}_k|$ is the number of agents in class $k \triangleq a, b$, with $N_a + N_b = N$. The empirical distribution of the agents sampled independently of the initial conditions and Wiener processes within populations \mathcal{A}^a and \mathcal{A}^b at time t_0 is denoted by $\pi^N = (\pi_a^N, \pi_b^N)$, where $\pi_a^N = \frac{N_a}{N}$ and $\pi_b^N = \frac{N_b}{N}$.

Assumption 6. There exists $\pi = (\pi_a, \pi_b)$ such that $\lim_{N \rightarrow \infty} \pi^N \stackrel{\text{a.s.}}{=} \pi$.

In the following we introduce the admissible sets of controls for each agent. By definition $\mathcal{F}_{i,t}$, $1 \leq i \leq N$, is the increasing family of null set augmented σ -fields generated by $(x_0(\tau), x_i(\tau); 0 \leq \tau \leq t)$, and $\mathcal{F}_{0,t}$ is the increasing family of null set augmented σ -fields generated by $(x_0(\tau); 0 \leq \tau \leq t)$. \mathcal{F}_t^N is the increasing family of σ -fields generated by the set $\{x_j(\tau), x_0(\tau); 0 \leq \tau \leq t, 1 \leq j \leq N\}$. The set of control actions $\mathcal{W}_g^{N,L}$ consists of linear feedback control actions adapted to $\{\mathcal{F}_t^N, t \in [0, T]\}$, $1 \leq N < \infty$.

Assumption 7 (Major Agent Linear Controls). For the major agent \mathcal{A}_0 the set of control inputs \mathcal{W}_0^L is defined to be the collection of linear feedback controls adapted to the filtration $\{\mathcal{F}_{0,t}, t \in [0, T]\}$.

Assumption 8 (Minor Agent Linear Controls). For the minor agent \mathcal{A}_i , $1 \leq i \leq N$, the set of control inputs \mathcal{W}_i^L is defined to be the collection of linear feedback controls adapted to the filtration $\{\mathcal{F}_{i,t}, t \in [0, t_s^i]\}$, $1 \leq i \leq N$.

4. Hybrid mean field game approach

Following the mean field game methodology with a major agent (Huang, 2010; Nourian & Caines, 2013), the hybrid MFG problem is first solved in the infinite population limit where the average term in the finite population dynamics and cost functional of each agent is replaced by its infinite population limit, i.e. the mean field, and the major agent and a generic minor agent \mathcal{A}_i only use local information (i.e. $\mathcal{F}_{0,t}$, $\mathcal{F}_{i,t}$, respectively). Then specializing to linear systems (see e.g. Huang (2010)), the major agent's state is extended with the mean field, while the minor agent's state is extended with the mean field and the major agent's state; this yields hybrid LQG optimal control problems (see Appendix A) for each agent linked only through the mean field and the major agent's state. Then the main results of Huang (2010) and Nourian and Caines (2013) are (i) the existence of infinite population best response strategies which yield the Nash equilibria, and (ii) the infinite population best response strategies using local information applied to the finite population system yield an ε -Nash equilibrium (see Theorem 3).

In this section, first, the hybrid evolution of the mean field is derived. Then the extended hybrid optimal control problems for the major agent and minor agents are formed and addressed in the infinite population case. Finally, Theorem 3 is presented which links the infinite population and finite population hybrid LQG MFG problem solutions.

4.1. Hybrid evolution of mean field

Following the LQG MFG methodology (Huang, 2010), the mean field is defined as the limit, when it exists, of the average of minor agents' states when the population size goes to infinity

$$\bar{x}^k(t) = \lim_{N_k \rightarrow \infty} x^{(N_k)}(t) = \lim_{N_k \rightarrow \infty} \frac{1}{N_k} \sum_{i=1}^{N_k} x_i(t),$$

where $k \triangleq a, b$, for the case considered in this paper. Now, the control strategy for each minor agent is considered to have the general linear state feedback form

$$u_i = L_1^k x_i + L_2^k x_0 + \sum_{j=1, j \neq i}^{N_i} L_3^k x_j + m_k, \quad i \in \mathcal{S}_k, \quad (29)$$

for bounded time-varying matrices L_1^k , L_2^k , L_3^k , and m_k of appropriate dimension (for notation brevity here and in the rest of the paper time arguments are dropped unless for clarity). Then the mean field dynamics, in q.m., is obtained by substituting (29) in the minor agents' dynamics (26), and taking the average over subpopulation \mathcal{A}^k , and then its limit as $N_k \rightarrow \infty$.

With the assignment of discrete states Q_j introduced in Section 3.2, the set of the mean field equations is given by

$$d\bar{x}^{Q_j} = \bar{A}^{Q_j} \bar{x}^{Q_j} dt + \bar{G}^{Q_j} x_0^{Q_j} dt + \bar{m}^{Q_j} dt, \quad j = 0, 1, 2, 3. \quad (30)$$

For $Q_0 = q_{0ab}$, $\bar{x}^{Q_0} = [\bar{x}_a^T, \bar{x}_b^T]^T$ consists of the mean field \bar{x}_a of the subpopulation \mathcal{A}^a , and the mean field \bar{x}_b of the subpopulation \mathcal{A}^b with $\pi^{Q_0} = \pi$. The matrices in (30) are then

$$\bar{A}^{Q_0} = \begin{bmatrix} \bar{A}_a \\ \bar{A}_b \end{bmatrix}, \quad \bar{G}^{Q_0} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_b \end{bmatrix}, \quad \bar{m}^{Q_0} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_b \end{bmatrix}, \quad (31)$$

where $\bar{A}_a, \bar{A}_b \in \mathbb{R}^{n \times 2n}$, $\bar{G}_a, \bar{G}_b \in \mathbb{R}^{n \times n}$, $\bar{m}_a, \bar{m}_b \in \mathbb{R}^n$. The above matrices shall be determined from the consistency equations discussed in Section 4.4.

In case (i) in Section 3.2 where $Q_1 = q_{0a}$, the mean field is defined as $\bar{x}^{q_{0a}} = [\bar{x}_a^T, \bar{x}_b^T]^T$, hence $\pi^{q_{0a}} = \pi$, and

$$\bar{A}^{q_{0a}} = \begin{bmatrix} \bar{A}_a \\ \bar{A}_b \end{bmatrix}, \quad \bar{G}^{q_{0a}} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_b \end{bmatrix}, \quad \bar{m}^{q_{0a}} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_b \end{bmatrix}. \quad (32)$$

For case (ii) where $Q_1 = q_{0a}$, $\bar{x}^{q_{0a}} = \bar{x}_a$, and hence $\pi^{q_{0a}} = 1$, and the matrices in (30) are given as

$$\bar{A}^{q_{0a}} = \bar{A}_a, \quad \bar{G}^{q_{0a}} = \bar{G}_a, \quad \bar{m}^{q_{0a}} = \bar{m}_a, \quad (33)$$

where $\bar{A}_a \in \mathbb{R}^{n \times n}$, $\bar{G}_a \in \mathbb{R}^{n \times n}$, $\bar{m}_a \in \mathbb{R}^n$.

For case (iii) where $Q_1 = q_{0b}$, $\bar{x}^{q_{0b}} = \bar{x}_b$, and hence $\pi^{q_{0b}} = 1$, and the matrices in (30) are given by

$$\bar{A}^{q_{0b}} = \bar{A}_b, \quad \bar{G}^{q_{0b}} = \bar{G}_b, \quad \bar{m}^{q_{0b}} = \bar{m}_b. \quad (34)$$

For case (I) in Section 3.2 where $Q_2 = q_{0a}$, the mean field is defined as $\bar{x}^{q_{0a}} = \bar{x}_a$, and hence $\pi^{q_{0a}} = 1$, and the matrices in (30) are given as

$$\bar{A}^{q_{0a}} = \bar{A}_a, \quad \bar{G}^{q_{0a}} = \bar{G}_a, \quad \bar{m}^{q_{0a}} = \bar{m}_a. \quad (35)$$

For case (II) where $Q_2 = q_{0b}$, $\bar{x}^{q_{0b}} = \bar{x}_b$, and hence $\pi^{q_{0b}} = 1$, and the matrices in (30) are given by

$$\bar{A}^{q_{0b}} = \bar{A}_b, \quad \bar{G}^{q_{0b}} = \bar{G}_b, \quad \bar{m}^{q_{0b}} = \bar{m}_b. \quad (36)$$

For case (III) where $Q_2 = q_1$, $\bar{x}^{q_1} = 0$, hence $\pi^{q_1} = 0$.

Finally, for $Q_3 = q_2$, $\bar{x}^{Q_3} = 0$, and as a result $\pi^{Q_3} = 0$.

4.2. Major agent: Infinite populations

4.2.1. Hybrid dynamics and cost

The extended hybrid dynamics of the major agent in the infinite population, i.e. the dynamics for x_0^{ex, Q_j} is given by

$$dx_0^{ex, Q_j} = (\mathbb{A}_0^{Q_j} x_0^{ex, Q_j} + \mathbb{M}_0^{Q_j} + \mathbb{B}_0^{Q_j} u_0^{Q_j}) dt + \mathbb{D}_0^{Q_j} dW_0^{Q_j}, \quad (37)$$

$0 \leq j \leq 3$, where the dynamical matrices are given by

$$\begin{aligned} \mathbb{A}_0^{Q_j} &= \begin{bmatrix} A_0^{Q_j} & \pi^{Q_j} \otimes F_0^{Q_j} \\ \bar{C}_0^{Q_j} & \bar{A}_0^{Q_j} \end{bmatrix}, \quad \mathbb{M}_0^{Q_j} = \begin{bmatrix} 0_{n \times 1} \\ \bar{m}_0^{Q_j} \end{bmatrix}, \quad \mathbb{B}_0^{Q_j} = \begin{bmatrix} B_0^{Q_j} \\ 0_{\bullet \times \bullet} \end{bmatrix}, \\ \mathbb{D}_0^{Q_j} &= \begin{bmatrix} D_0^{Q_j} & 0_{\bullet \times \bullet} \\ 0_{\bullet \times \bullet} & 0_{\bullet \times \bullet} \end{bmatrix}, \quad W_0^{Q_j} = \begin{bmatrix} w_0 \\ 0_{\bullet \times \bullet} \end{bmatrix}. \end{aligned} \quad (38)$$

In (38), $0_{\bullet \times \bullet}$ denotes a zero matrix of appropriate dimension, and \otimes denotes the Kronecker product.

The cost functional for the extended major agent's hybrid system is given by

$$\begin{aligned} J_0^\infty(u_0) &= \frac{1}{2} \mathbb{E} \left[\|x_0^{\text{ex}, Q_3}(T)\|_{\mathbb{P}_0^{Q_3}}^2 + \sum_{j=1}^3 \|x_0^{\text{ex}, Q_j}(t_j^-)\|_{\mathbb{C}_{0, \sigma_j}}^2 \right. \\ &\quad \left. + \sum_{j=0}^3 \int_{t_j}^{t_{j+1}} (\|x_0^{\text{ex}, Q_j}(s)\|_{\mathbb{P}_0^{Q_j}}^2 + \|u_0^{Q_j}(s)\|_{R_0^{Q_j}}^2) ds \right], \end{aligned} \quad (39)$$

where $t_0 = 0$, $t_4 = T$. In (39), the first term denotes terminal cost and the third term denotes running cost where the corresponding weight matrices are defined as

$$\begin{aligned} \bar{\mathbb{P}}_0^{Q_3} &= \bar{P}_0^2, \\ \mathbb{P}_0^{Q_j} &= [I_n, -\pi^{Q_j} \otimes H_0^{Q_j}]^T P_0^{Q_j} [I_n, -\pi^{Q_j} \otimes H_0^{Q_j}], \end{aligned} \quad (40)$$

where I_n denotes the identity matrix of size n . Moreover, the second term in (39) denotes the switching cost corresponding to the terminal cost of the quitting agents, for which the associated weight matrix \mathbb{C}_{0, σ_j} shall be identified for each switching in Section 4.2.2.

Now the dynamical and weight matrices introduced in their general form, respectively, in (38) and (40) are specified for each discrete state Q_j , $0 \leq j \leq 3$.

Over the interval $[t_0, t_1)$, and in discrete state Q_0 , the dynamics of the continuous state $x_0^{\text{ex}, Q_0} = [x_0^T, \bar{x}_a^T, \bar{x}_b^T]^T$ is determined by (37) with

$$\begin{aligned} \mathbb{A}_0^{Q_0} &= \begin{bmatrix} A_0^1 & \pi \otimes F_0^1 \\ \bar{C}_a & \bar{A}_a \\ \bar{C}_b & \bar{A}_b \end{bmatrix}, \quad \mathbb{M}_0^{Q_0} = \begin{bmatrix} 0_{n \times 1} \\ \bar{m}_a \\ \bar{m}_b \end{bmatrix}, \quad W_0^{Q_0} = \begin{bmatrix} w_0 \\ 0_{2r \times 1} \end{bmatrix}, \\ \mathbb{B}_0^{Q_0} &= \begin{bmatrix} B_0^1 \\ 0_{2n \times m} \end{bmatrix}, \quad \mathbb{D}_0^{Q_0} = \begin{bmatrix} D_0^1 & 0_{n \times 2r} \\ 0_{2n \times r} & 0_{2n \times 2r} \end{bmatrix}, \end{aligned} \quad (41)$$

where $\pi \otimes F_0^1 = [\pi_a F_0^1, \pi_b F_0^1]$, and $\mathbb{P}_0^{Q_0}$ in (39) is given by

$$\mathbb{P}_0^{Q_0} = [I_n, -\pi_a H_0^1, -\pi_b H_0^1]^T P_0^1 [I_n, -\pi_a H_0^1, -\pi_b H_0^1]. \quad (42)$$

We also define

$$\bar{\mathbb{P}}_0^{Q_0} = [I_n, -\pi_a H_0^1, -\pi_b H_0^1]^T \bar{P}_0^1 [I_n, -\pi_a H_0^1, -\pi_b H_0^1], \quad (43)$$

which will be used in Section 4.2.2 to specify the switching cost at t_1 .

Over the interval $[t_1, t_2)$, in case (i) where $Q_1 = q_{0ab}$ holds over the interval $[t_1, t_2)$, the dynamics of $x_0^{\text{ex}, q_{0ab}} = [x_0^T, \bar{x}_a^T, \bar{x}_b^T]^T$ is governed by (37) with

$$\begin{aligned} \mathbb{A}_0^{q_{0ab}} &= \begin{bmatrix} A_0^2 & \pi \otimes F_0^2 \\ \bar{C}_a & \bar{A}_a \\ \bar{C}_b & \bar{A}_b \end{bmatrix}, \quad \mathbb{M}_0^{q_{0ab}} = \begin{bmatrix} 0_{n \times 1} \\ \bar{m}_a \\ \bar{m}_b \end{bmatrix}, \\ \mathbb{B}_0^{q_{0ab}} &= \begin{bmatrix} B_0^2 \\ 0_{2n \times m} \end{bmatrix}, \quad \mathbb{D}_0^{q_{0ab}} = \begin{bmatrix} D_0^2 & 0_{n \times 2r} \\ 0_{2n \times r} & 0_{2n \times 2r} \end{bmatrix}, \quad W_0^{q_{0ab}} = \begin{bmatrix} w_0 \\ 0_{2r \times 1} \end{bmatrix}, \end{aligned} \quad (44)$$

and $\mathbb{P}_0^{q_{0ab}}$ in (39) and $\bar{\mathbb{P}}_0^{q_{0ab}}$ (to be used in Section 4.2.2 for specifying the switching cost at t_2 .) are given by

$$\mathbb{P}_0^{q_{0ab}} = [I_n, -\pi_a H_0^2, -\pi_b H_0^2]^T P_0^2 [I_n, -\pi_a H_0^2, -\pi_b H_0^2], \quad (45)$$

$$\bar{\mathbb{P}}_0^{q_{0ab}} = [I_n, -\pi_a H_0^2, -\pi_b H_0^2]^T \bar{P}_0^2 [I_n, -\pi_a H_0^2, -\pi_b H_0^2]. \quad (46)$$

Over the interval $[t_1, t_2)$, in case (ii) where $Q_1 = q_{1a}$ holds, the dynamics and cost functional for $x_0^{\text{ex}, q_{1a}} = [x_0^T, \bar{x}_a^T]^T$ are, respectively, determined by (37) and (39) with

$$\begin{aligned} \mathbb{A}_0^{q_{1a}} &= \begin{bmatrix} A_0^1 & F_0^1 \\ \bar{C}_a & \bar{A}_a \end{bmatrix}, \quad \mathbb{M}_0^{q_{1a}} = \begin{bmatrix} 0_{n \times 1} \\ \bar{m}_a \end{bmatrix}, \quad \mathbb{B}_0^{q_{1a}} = \begin{bmatrix} B_0^1 \\ 0_{n \times m} \end{bmatrix}, \\ \mathbb{D}_0^{q_{1a}} &= \begin{bmatrix} D_0^1 & 0_{n \times r} \\ 0_{n \times r} & 0_{n \times r} \end{bmatrix}, \quad W_0^{q_{1a}} = \begin{bmatrix} w_0 \\ 0_{r \times 1} \end{bmatrix}, \end{aligned} \quad (47)$$

$$\mathbb{P}_0^{q_{1a}} = [I_n, -H_0^1]^T P_0^1 [I_n, -H_0^1], \quad (48)$$

$$\bar{\mathbb{P}}_0^{q_{1a}} = [I_n, -H_0^1]^T \bar{P}_0^1 [I_n, -H_0^1]. \quad (49)$$

Over the interval $[t_1, t_2)$, in case (iii) where $Q_1 = q_{1b}$ holds, $x^{\text{ex}, q_{1b}} = [x_0^T, \bar{x}_b^T]^T$ and

$$\mathbb{A}_0^{q_{1b}} = \begin{bmatrix} A_0^1 & F_0^1 \\ \bar{C}_b & \bar{A}_b \end{bmatrix}, \quad \mathbb{M}_0^{q_{1b}} = \begin{bmatrix} 0_{n \times m} \\ \bar{m}_b \end{bmatrix}, \quad \mathbb{B}_0^{q_{1b}} = \begin{bmatrix} B_0^1 \\ 0_{n \times m} \end{bmatrix},$$

$$\mathbb{D}_0^{q_{1b}} = \begin{bmatrix} D_0^1 & 0_{n \times r} \\ 0_{n \times r} & 0_{n \times r} \end{bmatrix}, \quad W_0^{q_{1b}} = \begin{bmatrix} w_0 \\ 0_{r \times 1} \end{bmatrix}, \quad (50)$$

$$\mathbb{P}_0^{q_{1b}} = [I_n, -H_0^1]^T P_0^1 [I_n, -H_0^1], \quad (51)$$

$$\bar{\mathbb{P}}_0^{q_{1b}} = [I_n, -H_0^1]^T \bar{P}_0^1 [I_n, -H_0^1]. \quad (52)$$

Over the interval $[t_2, t_3)$, in case (I) where $Q_2 = q_{2a}$ holds, $x^{\text{ex}, q_{2a}} = [x_0^T, \bar{x}_a^T]^T$ and

$$\mathbb{A}_0^{q_{2a}} = \begin{bmatrix} A_0^2 & F_0^2 \\ \bar{C}_a & \bar{A}_a \end{bmatrix}, \quad \mathbb{M}_0^{q_{2a}} = \begin{bmatrix} 0_{n \times 1} \\ \bar{m}_a \end{bmatrix}, \quad \mathbb{B}_0^{q_{2a}} = \begin{bmatrix} B_0^2 \\ 0_{n \times m} \end{bmatrix},$$

$$\mathbb{D}_0^{q_{2a}} = \begin{bmatrix} D_0^2 & 0_{n \times r} \\ 0_{n \times r} & 0_{n \times r} \end{bmatrix}, \quad W_0^{q_{2a}} = \begin{bmatrix} w_0 \\ 0_{r \times 1} \end{bmatrix}, \quad (53)$$

$$\mathbb{P}_0^{q_{2a}} = [I_n, -H_0^2]^T P_0^2 [I_n, -H_0^2], \quad (54)$$

$$\bar{\mathbb{P}}_0^{q_{2a}} = [I_n, -H_0^2]^T \bar{P}_0^2 [I_n, -H_0^2]. \quad (55)$$

Over the interval $[t_2, t_3)$, in case (II) where $Q_2 = q_{2b}$ holds, $x^{\text{ex}, q_{2b}} = [x_0^T, \bar{x}_b^T]^T$ and

$$\mathbb{A}_0^{q_{2b}} = \begin{bmatrix} A_0^2 & F_0^2 \\ \bar{C}_b & \bar{A}_b \end{bmatrix}, \quad \mathbb{M}_0^{q_{2b}} = \begin{bmatrix} 0_{n \times 1} \\ \bar{m}_b \end{bmatrix}, \quad \mathbb{B}_0^{q_{2b}} = \begin{bmatrix} B_0^2 \\ 0_{n \times m} \end{bmatrix},$$

$$\mathbb{D}_0^{q_{2b}} = \begin{bmatrix} D_0^2 & 0_{n \times r} \\ 0_{n \times r} & 0_{n \times r} \end{bmatrix}, \quad W_0^{q_{2b}} = \begin{bmatrix} w_0 \\ 0_{r \times 1} \end{bmatrix}, \quad (56)$$

$$\mathbb{P}_0^{q_{2b}} = [I_n, -H_0^2]^T P_0^2 [I_n, -H_0^2], \quad (57)$$

$$\bar{\mathbb{P}}_0^{q_{2b}} = [I_n, -H_0^2]^T \bar{P}_0^2 [I_n, -H_0^2]. \quad (58)$$

Over the interval $[t_2, t_3)$, in case (III) where $Q_2 = q_1$ holds, $x^{\text{ex}, q_1} = x_0$ and

$$\mathbb{A}_0^{q_1} = A_0^1, \quad \mathbb{M}_0^{q_1} = 0_{n \times 1}, \quad \mathbb{B}_0^{q_1} = B_0^1, \quad \mathbb{D}_0^{q_1} = D_0^1,$$

$$W_0^{q_1} = w_0, \quad \mathbb{P}_0^{q_1} = P_0^1, \quad \bar{\mathbb{P}}_0^{q_1} = \bar{P}_0^1.$$

Finally, over the interval $[t_3, T]$, in discrete state Q_3 , $x^{\text{ex}, Q_3} = x_0$ and

$$\mathbb{A}_0^{Q_3} = A_0^2, \quad \mathbb{M}_0^{Q_3} = 0_{n \times 1}, \quad \mathbb{B}_0^{Q_3} = B_0^2, \quad \mathbb{D}_0^{Q_3} = D_0^2,$$

$$W_0^{Q_3} = w_0, \quad \mathbb{P}_0^{Q_3} = P_0^2, \quad \bar{\mathbb{P}}_0^{Q_3} = \bar{P}_0^2.$$

4.2.2. Jump transition maps and switching costs

The major agent's switching cost associated with t_j takes into account the cost incurred when a change occurs in the system. To

identify it, we define the notation $M_0^{Q_j}(l : m)$, $0 \leq j \leq 3$, which is formed by using matrix $\mathbb{P}_0^{Q_j}$ wherein all the entries are made zero except those associated with its l th to m th columns and rows. Hence it has the same dimension (size) as $\mathbb{P}_0^{Q_j}$, i.e.

$$M_0^{Q_j}(l : m) = \left[\begin{array}{c|c|c} \overbrace{0 \quad 0 \quad 0}^{\mathbb{P}_0^{Q_j}(:, l : m)} \\ \hline \mathbb{P}_0^{Q_j}(l : m, :) \\ \hline 0 \quad 0 \quad 0 \end{array} \right] \Bigg\}_{\substack{Q_j \\ \text{size}(\mathbb{P}_0^{Q_j})}} \quad (59)$$

where $\mathbb{P}_0^{Q_j}(:, l : m)$ and $\mathbb{P}_0^{Q_j}(l : m, :)$, respectively, denote l th to m th columns and rows of $\mathbb{P}_0^{Q_j}$.

The values of the major agent's continuous state before and after switching at t_1 satisfy the following jump map

$$x_0^{ex, Q_1}(t_1) = \Psi_{0, \sigma_1} x_0^{ex, Q_0}(t_1 -). \quad (60)$$

For the transition between Q_0 and case (i) for Q_1 where $Q_1 = q_{2ab}$ the map Ψ_{0, σ_1} is the identity matrix, i.e.

$$\Psi_{0, \sigma_1} = \Psi_{0, \sigma_{q_{1ab}, q_{2ab}}} = I_{3n}. \quad (61)$$

This transition is not accompanied by change in the dimension of the major agent's extended state. Furthermore, the weight matrix for the corresponding switching cost is

$$\mathbb{C}_{0, \sigma_1} = \mathbb{C}_{0, \sigma_{q_{1ab}, q_{2ab}}} = \mathbf{0}_{3n \times 3n}. \quad (62)$$

For the transition between Q_0 and case (ii) where $Q_1 = q_{1a}$

$$\Psi_{0, \sigma_1} = \Psi_{0, \sigma_{q_{1ab}, q_{1a}}} = \begin{bmatrix} I_n & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & I_n & \mathbf{0}_{n \times n} \end{bmatrix}, \quad (63)$$

$$\mathbb{C}_{0, \sigma_1} = \mathbb{C}_{0, \sigma_{q_{1ab}, q_{1a}}} = M_0^{q_{1ab}}(2n + 1 : 3n). \quad (64)$$

For the transition between Q_0 and case (iii) where $Q_1 = q_{1b}$

$$\Psi_{0, \sigma_1} = \Psi_{0, \sigma_{q_{1ab}, q_{1b}}} = \begin{bmatrix} I_n & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & I_n \end{bmatrix}, \quad (65)$$

$$\mathbb{C}_{0, \sigma_1} = \mathbb{C}_{0, \sigma_{q_{1ab}, q_{1b}}} = M_0^{q_{1ab}}(n + 1 : 2n). \quad (66)$$

The values of the major agent's continuous state before and after the switching at t_2 satisfy the jump transition map

$$x_0^{ex, Q_2}(t_2) = \Psi_{0, \sigma_2} x_0^{ex, Q_1}(t_2 -), \quad (67)$$

where

$$\Psi_{0, \sigma_2} = \begin{cases} \Psi_{0, \sigma_{q_{1a}, q_{2a}}} = I_{2n}, \\ \Psi_{0, \sigma_{q_{1a}, q_{1a}}} = [I_n \quad \mathbf{0}_{n \times n}], \\ \Psi_{0, \sigma_{q_{2ab}, q_{2a}}} = \begin{bmatrix} I_n & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & I_n & \mathbf{0}_{n \times n} \end{bmatrix}, \\ \Psi_{0, \sigma_{q_{2ab}, q_{2b}}} = \begin{bmatrix} I_n & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & I_n \end{bmatrix}, \\ \Psi_{0, \sigma_{q_{1b}, q_{2b}}} = I_{2n \times 2n}, \\ \Psi_{0, \sigma_{q_{1b}, q_{1a}}} = [I_n \quad \mathbf{0}_{n \times n}], \end{cases} \quad (68)$$

Furthermore, the matrix coefficient \mathbb{C}_{0, σ_2} of the switching cost at t_2 for each case is defined as

$$\mathbb{C}_{0, \sigma_2} = \begin{cases} \mathbb{C}_{0, \sigma_{q_{1a}, q_{2a}}} = \mathbf{0}_{2n \times 2n}, \\ \mathbb{C}_{0, \sigma_{q_{1a}, q_{1a}}} = M_0^{q_{1a}}(n + 1 : 2n), \\ \mathbb{C}_{0, \sigma_{q_{2ab}, q_{2a}}} = M_0^{q_{2ab}}(2n + 1 : 3n), \\ \mathbb{C}_{0, \sigma_{q_{2ab}, q_{2b}}} = M_0^{q_{2ab}}(n + 1 : 2n), \\ \mathbb{C}_{0, \sigma_{q_{1b}, q_{2b}}} = \mathbf{0}_{2n \times 2n}, \\ \mathbb{C}_{0, \sigma_{q_{1b}, q_{1a}}} = M_0^{q_{1b}}(n + 1 : 2n). \end{cases} \quad (69)$$

The values of the major agent's continuous state before and after the switching at t_3 satisfy the following jump map

$$x_0^{ex, Q_3}(t_3) = \Psi_{0, \sigma_3} x_0^{ex, Q_2}(t_3 -), \quad (70)$$

$$\Psi_{0, \sigma_3} = \begin{cases} \Psi_{0, \sigma_{q_{2a}, q_{2a}}} = [I_n \quad \mathbf{0}_{n \times n}], \\ \Psi_{0, \sigma_{q_{1a}, q_{2a}}} = I_n, \\ \Psi_{0, \sigma_{q_{2b}, q_{2a}}} = [I_n \quad \mathbf{0}_{n \times n}]. \end{cases} \quad (71)$$

Accordingly, the matrix coefficient \mathbb{C}_{0, σ_3} of the switching cost at t_3 for each case is given by

$$\mathbb{C}_{0, \sigma_3} = \begin{cases} \mathbb{C}_{0, \sigma_{q_{2a}, q_{2a}}} = M_0^{q_{2a}}(n + 1 : 2n), \\ \mathbb{C}_{0, \sigma_{q_{1a}, q_{2a}}} = \mathbf{0}_{n \times n}, \\ \mathbb{C}_{0, \sigma_{q_{2b}, q_{2a}}} = M_0^{q_{2b}}(n + 1 : 2n). \end{cases} \quad (72)$$

Notice that some of the transitions of (60), (67), (70) are between the spaces of the same dimension such as (61) while other transitions may be accompanied by changes in the dimension of the state space, e.g. (63) is a mapping from \mathbb{R}^{3n} into \mathbb{R}^{2n} . These dimension changes are permitted in the stochastic hybrid systems framework of Pakniyat and Caines (2016, 2017a) (see Pakniyat and Caines (2017b) for another motivating example for change of dimension at switching).

4.2.3. Best response hybrid control action

To obtain the best response hybrid control action for the major agent in the infinite population, we utilize Theorem 1 in Section 2 developed for single agent hybrid LQG optimal control problems. By the definition of the terms $\mathbb{D}_0^{Q_j}$, they automatically satisfy the condition (3) (see Section 2), or equivalently condition A1 in Pakniyat and Caines (2016, Eq. (3)) as

$$\mathbb{D}_0^{Q_j} = \Psi_{0, \sigma_j} \mathbb{D}_0^{Q_{j-1}}, \quad j = 1, 2, 3, \quad (73)$$

holds for all the jump transition maps introduced in this section. Moreover, it is assumed conditions (8)–(10) in Section 2 hold. Therefore, the optimal controlled switching times for the major agent are \mathcal{F}_t -independent. Then an application of the hybrid LQG optimal control theory (see Theorem 1) yields the infinite population best response hybrid control action for discrete states $\{Q_0, \dots, Q_3\}$ as in

$$u_0^{Q_j, \circ}(t) = -[R_0^{Q_j}]^{-1} [\mathbb{B}_0^{Q_j}]^T \Pi_0^{Q_j}(t) x_0^{ex, Q_j}(t), \quad (74)$$

$$-\dot{\Pi}_0^{Q_j} = \Pi_0^{Q_j} \mathbb{A}_0^{Q_j} + [\mathbb{A}_0^{Q_j}]^T \Pi_0^{Q_j} - \Pi_0^{Q_j} \mathbb{B}_0^{Q_j} [R_0^{Q_j}]^{-1} [\mathbb{B}_0^{Q_j}]^T \Pi_0^{Q_j} + \mathbb{P}_0^{Q_j}, \quad (75)$$

subject to the terminal and boundary conditions

$$\Pi_0^{Q_3}(T) = \mathbb{P}_0^{Q_3}, \quad (76)$$

$$\Pi_0^{Q_j-1}(t_j) = \Psi_{0,\sigma_j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,\sigma_j} + \mathbb{C}_{0,\sigma_j}, \quad (77)$$

$$\begin{aligned} & \mathbb{P}_0^{Q_j-1} + \Pi_0^{Q_j-1}(t_j) \mathbb{A}_0^{Q_j-1} + [\mathbb{A}_0^{Q_j-1}]^T \Pi_0^{Q_j-1}(t_j) \\ & - \Pi_0^{Q_j-1}(t_j) \mathbb{B}_0^{Q_j-1} [R_0^{Q_j-1}]^{-1} [\mathbb{B}_0^{Q_j-1}]^T \Pi_0^{Q_j-1}(t_j) \\ & = \Psi_{0,\sigma_j}^T \left(\mathbb{P}_0^{Q_j} + \Pi_0^{Q_j}(t_j) \mathbb{A}_0^{Q_j} + [\mathbb{A}_0^{Q_j}]^T \Pi_0^{Q_j}(t_j) \right. \\ & \left. - \Pi_0^{Q_j}(t_j) \mathbb{B}_0^{Q_j} [R_0^{Q_j}]^{-1} [\mathbb{B}_0^{Q_j}]^T \Pi_0^{Q_j}(t_j) \right) \Psi_{0,\sigma_j} + \frac{\partial \mathbb{C}_{0,\sigma_j}}{\partial t} \Big|_{t=t_j}, \quad (78) \end{aligned}$$

where $t_j, j = 1, 2, 3$, indicate the times of change in the system due to the major agent's switching of dynamics or cessation of subpopulations of minor agents.

4.3. Minor agents: Infinite population

4.3.1. Hybrid dynamics and costs

The extended dynamics for a generic minor agent $\mathcal{A}_i^k, 1 \leq i \leq N$, in the population $k \triangleq a, b$, with the extended state x_i^{ex,Q_j} has a general form as in

$$dx_i^{ex,Q_j} = (\mathbb{A}_k^{Q_j} x_i^{ex,Q_j} + \mathbb{M}_k^{Q_j} + \mathbb{B}_k^{Q_j} u_i^{Q_j}) dt + \mathbb{D}_k^{Q_j} dW_i^{Q_j}, \quad (79)$$

where

$$\begin{aligned} \mathbb{A}_k^{Q_j} &= \begin{bmatrix} A_k & [G_k \ \pi^{Q_j} \otimes F_k] \\ \mathbf{0}_{\bullet \times \bullet} & \mathbb{A}_0^{Q_j} - \mathbb{B}_0^{Q_j} R_0^{-1} [\mathbb{B}_0^{Q_j}]^T \Pi_0^{Q_j} \end{bmatrix}, \quad \mathbb{M}_k^{Q_j} = \begin{bmatrix} \mathbf{0}_{n \times 1} \\ \mathbb{M}_0^{Q_j} \end{bmatrix}, \\ \mathbb{B}_k^{Q_j} &= \begin{bmatrix} B_k \\ \mathbf{0}_{\bullet \times \bullet} \end{bmatrix}, \quad \mathbb{D}_k^{Q_j} = \begin{bmatrix} D_k & \mathbf{0}_{\bullet \times \bullet} \\ \mathbf{0}_{\bullet \times \bullet} & \mathbb{D}_0^{Q_j} \end{bmatrix}, \quad W_i^{Q_j} = \begin{bmatrix} w_i \\ w_0 \end{bmatrix}. \quad (80) \end{aligned}$$

Notice that in (79) the major agent's closed-loop dynamics at discrete state $Q_j, 0 \leq j \leq 3$, given by (37) is used to derive the extended dynamics for minor agent \mathcal{A}_i^k at discrete state $Q_j, 0 \leq j \leq 3$. Similar to the major agent's case, $\mathbf{0}_{\bullet \times \bullet}$ in (80) denotes a zero matrix of appropriate dimension.

The cost functional for the extended minor agent \mathcal{A}_i^k 's hybrid system is given by

$$\begin{aligned} J_i^{\infty,k}(u_i, u_0) &= \frac{1}{2} \mathbb{E} \left[\|x_i^{ex,Q_\ell}(t_s^k)\|_{\mathbb{P}_k^{Q_\ell}}^2 + \sum_{j=1}^{\ell} \|x_i^{ex,Q_j}(t_j^-)\|_{\mathbb{C}_{i,\sigma_j}^k}^2 \right. \\ & \left. + \sum_{j=0}^{\ell} \int_{t_j}^{t_{j+1}} (\|x_i^{ex,Q_j}(s)\|_{\mathbb{Q}_j}^2 + \|u_i^{Q_j}(s)\|_{R_k}^2) ds \right], \quad (81) \end{aligned}$$

where Q_ℓ denotes the discrete state after which minor agent \mathcal{A}_i^k quits the system at the individual stopping time t_s^k and $\ell \in \{0, 1, 2\}$ denotes the index of the associate discrete state. The weight matrices associated with the terminal cost (first term) and the running cost (third term) in (81) are, respectively, given by

$$\begin{aligned} \bar{\mathbb{P}}_k^{Q_\ell} &= \bar{P}_k, \\ \mathbb{P}_k^{Q_j} &= [I_n, -H_1^k, -\pi^{Q_j} \otimes H_2^k]^T P_k [I_n, -H_1^k, -\pi^{Q_j} \otimes H_2^k], \\ \bar{\mathbb{P}}_k^{Q_j} &= [I_n, -H_1^k, -\pi^{Q_j} \otimes H_2^k]^T \bar{P}_k [I_n, -H_1^k, -\pi^{Q_j} \otimes H_2^k], \quad (82) \end{aligned}$$

where $\bar{\mathbb{P}}_k^{Q_j}$ shall be used to specify the weight matrix $\mathbb{C}_{i,\sigma_j}^k$, associated with the switching cost (second term) in (81) in a similar manner to that of the major agent in Section 4.2.2. The values of minor agent \mathcal{A}_i^k continuous state before and after the switching time t_j satisfy the following jump transition map

$$x_i^{ex,Q_j}(t_j) = \Psi_{i,\sigma_j}^k x_i^{ex,Q_{j-1}}(t_j^-). \quad (83)$$

The realizations of $\mathbb{C}_{i,\sigma_j}^k$ and Ψ_{i,σ_j}^k associated with the switching times $t_j, j = 1, 2, 3$, are presented in Appendix B.

4.3.2. Best response hybrid control actions

The optimal stopping problem for a minor agent is equivalent to a hybrid optimal control problem in which the dynamics and costs become zero after stopping. Let us assume that minor agent \mathcal{A}_i^k stops at time t_s^k after the discrete state $Q_\ell, \ell \in \{0, 1, 2\}$. The definitions for $\mathbb{D}_k^{Q_j}$ directly result in the satisfaction of condition (3) (see Section 2), or equivalently condition A1 in Pakniyat and Caines (2016, Eq. (3)), i.e.

$$\mathbb{D}_k^{Q_j} = \Psi_{i,\sigma_j}^k \mathbb{D}_k^{Q_{j-1}}, \quad j \in \{1, \dots, \ell\}, \quad k \triangleq a, b. \quad (84)$$

Furthermore, it is assumed in this paper that conditions (8)–(10), and (16)–(18), respectively, hold on the parameters $\mathbb{A}_k, \mathbb{B}_k, \mathbb{D}_k, \bar{\mathbb{P}}_k, \mathbb{P}_k, \Psi_{i,\sigma}^k$, etc., of the extended system and, consequently, the optimal stopping times t_s^k and switching times t_j exist and are determined by the solutions to (8)–(10), and (16)–(18). Moreover, the optimal stopping and switching times are \mathcal{F}_t -independent and depend only on the dynamical parameters; this implies that all minor agents of the same type switch and stop at the same instant. Then the application of the results of Theorem 1 and Corollary 2 yields the infinite population best response strategies for the discrete states $\{Q_0, \dots, Q_\ell\}$ given by

$$u_i^{Q_j,o}(t) = -R_k^{-1} [\mathbb{B}_k^{Q_j}]^T \Pi_k^{Q_j}(t) x_i^{ex,Q_j}(t), \quad (85)$$

with

$$-\dot{\Pi}_k^{Q_j} = \Pi_k^{Q_j} \mathbb{A}_k^{Q_j} + \mathbb{A}_{k,Q_j}^T \Pi_k^{Q_j} - \Pi_k^{Q_j} \mathbb{B}_k^{Q_j} R_k^{-1} [\mathbb{B}_k^{Q_j}]^T \Pi_k^{Q_j} + \mathbb{P}_k^{Q_j} \quad (86)$$

subject to the terminal conditions

$$\Pi_k^{Q_\ell}(t_s^k) = \bar{\mathbb{P}}_k^{Q_\ell}, \quad (87)$$

$$\begin{aligned} & \left(\mathbb{P}_k^{Q_\ell} + \bar{\mathbb{P}}_k^{Q_\ell} \mathbb{A}_k^{Q_\ell} + \mathbb{A}_{k,Q_\ell}^T \bar{\mathbb{P}}_k^{Q_\ell} - \bar{\mathbb{P}}_k^{Q_\ell} \mathbb{B}_k^{Q_\ell} R_k^{-1} [\mathbb{B}_k^{Q_\ell}]^T \bar{\mathbb{P}}_k^{Q_\ell} \right) \Big|_{t=t_s^k} \\ & = \frac{\partial \mathbb{C}_{i,\sigma_\ell}^k}{\partial t} \Big|_{t=t_s^k}, \quad (88) \end{aligned}$$

and the boundary conditions

$$\Pi_k^{Q_{j-1}}(t_j) = [\Psi_{i,\sigma_j}^k]^T \Pi_k^{Q_j}(t_j) \Psi_{i,\sigma_j}^k + \mathbb{C}_{i,\sigma_j}^k, \quad (89)$$

$$\begin{aligned} & \mathbb{P}_k^{Q_{j-1}} + \Pi_k^{Q_{j-1}}(t_j) \mathbb{A}_k^{Q_{j-1}} + [\mathbb{A}_k^{Q_{j-1}}]^T \Pi_k^{Q_{j-1}}(t_j) \\ & - \Pi_k^{Q_{j-1}}(t_j) \mathbb{B}_k^{Q_{j-1}} R_k^{-1} [\mathbb{B}_k^{Q_{j-1}}]^T \Pi_k^{Q_{j-1}}(t_j) \\ & = [\Psi_{i,\sigma_j}^k]^T \left(\mathbb{P}_k^{Q_j} + \Pi_k^{Q_j}(t_j) \mathbb{A}_k^{Q_j} + [\mathbb{A}_k^{Q_j}]^T \Pi_k^{Q_j}(t_j) \right. \\ & \left. - \Pi_k^{Q_j}(t_j) \mathbb{B}_k^{Q_j} R_k^{-1} [\mathbb{B}_k^{Q_j}]^T \Pi_k^{Q_j}(t_j) \right) \Psi_{i,\sigma_j}^k + \frac{\partial \mathbb{C}_{i,\sigma_j}^k}{\partial t} \Big|_{t=t_j}, \quad (90) \end{aligned}$$

if $t_j < t_s^k$, where $\{t_j, j \in \{1, \dots, \ell\}\}$ indicate the times of change in the system due to the major agent's switching of dynamics or cessation of the other subpopulation of minor agents. We observe that for the case where the subpopulation $k, k \triangleq a, b$, stops at time t_1 , the boundary conditions (89) and (90) are irrelevant for the agents belonging to the quitting subpopulation.

4.4. Hybrid mean field consistency equations and ε -Nash equilibrium

Let us define

$$\Pi_k^{Q_j} = \begin{bmatrix} \Pi_{k,11}^{Q_j} & \Pi_{k,12}^{Q_j} & \Pi_{k,13}^{Q_j} \\ \Pi_{k,21}^{Q_j} & \Pi_{k,22}^{Q_j} & \Pi_{k,23}^{Q_j} \\ \Pi_{k,31}^{Q_j} & \Pi_{k,32}^{Q_j} & \Pi_{k,33}^{Q_j} \end{bmatrix}, \quad k \triangleq a, b,$$

$$\mathbf{e}_k^{Q_j} = \begin{cases} I_n & \text{if } \bar{x}^{Q_j} = \bar{x}_k, \\ [I_n, \mathbf{0}_{n \times n}] & \text{if } \{\bar{x}^{Q_j} \neq \bar{x}_k\} \wedge \{k = a\}, \\ [\mathbf{0}_{n \times n}, I_n] & \text{if } \{\bar{x}^{Q_j} \neq \bar{x}_k\} \wedge \{k = b\}. \end{cases} \quad (91)$$

We substitute the set of obtained best response strategies (85) in (26). Then we take an average over the corresponding sub-population and take its limit as $N_t \rightarrow \infty$. In order to generate a mean field game equilibrium the obtained equation and (30) must correspond to the same dynamical system generating the mean field. Consequently we obtain the Consistency Condition equations, determining \bar{A}^{Q_j} , \bar{G}^{Q_j} , \bar{m}^{Q_j} given by

$$\begin{aligned} \bar{\Pi}_0^{Q_j} &= \Pi_0^{Q_j} \mathbb{B}_0^{Q_j} R_0^{-1} [\mathbb{B}_0^{Q_j}]^T \Pi_0^{Q_j} - \Pi_0^{Q_j} \mathbb{A}_0^{Q_j} - [\mathbb{A}_0^{Q_j}]^T \Pi_0^{Q_j} - \mathbb{P}_0^{Q_j}, \\ \bar{\Pi}_k^{Q_j} &= \Pi_k^{Q_j} \mathbb{B}_k^{Q_j} R_k^{-1} [\mathbb{B}_k^{Q_j}]^T \Pi_k^{Q_j} - \Pi_k^{Q_j} \mathbb{A}_k^{Q_j} - [\mathbb{A}_k^{Q_j}]^T \Pi_k^{Q_j} - \mathbb{P}_k^{Q_j}, \\ \bar{A}_k^{Q_j} &= [A_k - B_k R_k^{-1} B_k^T \Pi_{k,11}^{Q_j}] \mathbf{e}_k^{Q_j} + \pi^{Q_j} \otimes F_k - B_k R_k^{-1} B_k^T \Pi_{k,13}^{Q_j}, \\ \bar{G}_k^{Q_j} &= G_k - B_k R_k^{-1} B_k^T \Pi_{k,12}^{Q_j}, \\ \bar{m}_k^{Q_j} &= 0, \end{aligned} \quad (92)$$

for each discrete state Q_j , $0 \leq j \leq 3$, and the corresponding population k , $k \triangleq a, b$. The set of Eqs. (92) forms a fixed-point problem for each discrete state Q_j , $1 \leq j \leq 3$, that should be solved by each minor agent in order to compute the matrices in the mean field dynamics.

Assumption 9. The parameters in (22)–(27) belong to a non-empty set which yields the existence and uniqueness of the solutions $(\Pi_0^{Q_j}, \Pi_k^{Q_j}, \bar{A}_k^{Q_j}, \bar{G}_k^{Q_j}, \bar{m}_k^{Q_j})$ to the resulting set of mean field fixed-point Eqs. (92) at each discrete state Q^j , $0 \leq j \leq 3$.

The following theorem links the infinite population equilibria to the finite population case.

Theorem 3 (ε -Nash Equilibrium for Hybrid LQG MFGs). *Subject to Assumptions 1–9, and subject to the assumption that the Eqs. (8)–(10), and (16)–(18) are satisfied, the \mathcal{F}_t -invariant optimal switching and stopping times t_1, t_2, t_3 exist and are uniquely determined. In that case system Eqs. (37)–(40), (79)–(82), together with the set of mean field Eqs. (92) generate a set of control laws which yields the infinite population Nash equilibrium. When the set of infinite population control laws $\mathcal{U}_{MF}^{N_t} \triangleq \{u_i^{Q_j, \circ}; 0 \leq i \leq N_t\}$, $0 \leq j \leq 3$, $1 \leq N_t \leq N < \infty$, given by (74), (85) is applied to the finite population system (22), (26), it yields an ε -Nash equilibrium for all ε , i.e. for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $N \geq N(\varepsilon)$:*

$$\begin{aligned} J_i^{Q_j, N}(u_i^{Q_j, \circ}, u_{-i}^{Q_j, \circ}) - \varepsilon &\leq \inf_{u_i \in \mathcal{U}_{MF}^{N, L}} J_i^{Q_j, N}(u_i, u_{-i}^{Q_j, \circ}) \\ &\leq J_i^{Q_j, N}(u_i^{Q_j, \circ}, u_{-i}^{Q_j, \circ}), \end{aligned}$$

where $J_i^{Q_j, N}(\cdot, \cdot)$ denotes the finite-population cost functional of the generic agent \mathcal{A}_i at the discrete state Q_j . \square

Proof. See Appendix C.

4.5. Hybrid dynamic programming methodology

The order of the switching and stopping events Q_0, Q_1, Q_2, Q_3 , if all of them occur, is assumed to be fixed. As depicted in Fig. 1 and explained in Section 3.2, there are three possible realizations for each of the discrete states Q_1 and Q_2 . The optimal sequence of switching, that is to say the discrete trajectory of the system, is determined via dynamic programming backward propagation. For this purpose, the steps below are followed.

Step 1. (Solving backwards for transitions from Q_3 to Q_2). Eq. (75) is solved for $\Pi_0^{Q_3}(t)$ backward in time, subject to the terminal condition (76). Then the values for $\Pi_0^{Q_3}(t)$ are substituted in the right hand side of (77) to obtain $\Pi_0^{Q_2}(t)$ for all three realizations of Ψ_{0, σ_3} and \mathbb{C}_{0, σ_3} given by (71) and (72), respectively. Next,

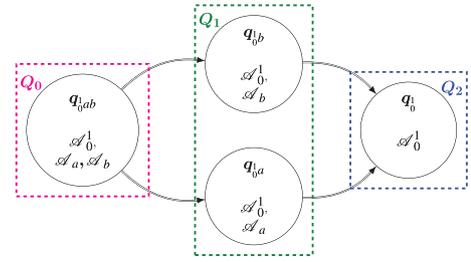


Fig. 2. Hybrid Automata Diagram with a single major agent and two populations of minor agents with stopping times.

we substitute $\Pi_0^{Q_2}(t)$ and the corresponding Ψ_{0, σ_3} and \mathbb{C}_{0, σ_3} in (78). Then the time instant at which (78) holds determines t_3 for the transition from the corresponding realization of Q_2 to Q_3 . The transitions from $Q_2 \triangleq q_{2b}$ to Q_3 or from $Q_2 \triangleq q_{2a}$ to Q_3 are equivalent to the stopping of subpopulation \mathcal{A}_b or \mathcal{A}_a , respectively, at the obtained switching time t_3 . Hence Eq. (88) must also hold at the associated t_3 for each of the mentioned cases. Similarly, for the transition from $Q_2 \triangleq q_1$ to Q_3 both (78) and (90) must hold at the same time.

We observe that if (78) does not hold for any of the realizations of $Q_2 = \{q_{2a}, q_1, q_{2b}\}$, then we conclude that Q_3 is not the final discrete state of the system. Subsequently, we start from Step 2 solving the dynamic programming backward in time from $t = T$.

Step 2. (Solving backwards for transitions from Q_2 to Q_1). Starting from the obtained realizations of Q_2 in Step 1 and the corresponding switching times t_3 , we follow a similar approach as in Step 1 to determine the realizations of Q_1 which may take place and their corresponding switching times t_2 . More specifically, Eq. (75) is solved with the boundary (terminal) condition (77) with $j = 3$ at t_3 . Then, for example, to determine from $Q_2 \triangleq q_{2b}$ which of (either of or neither of) the transitions to $Q_1 \triangleq q_{2ab}$ and $Q_1 \triangleq q_{1b}$ may occur, Eqs. (78), (88) and (78), (90) are checked, respectively.

Step 3. (Solving backwards for transitions from Q_1 to Q_0). Similar to the previous steps, starting from the cases for Q_1 and the value for t_2 determined in Step 2, it is investigated using Eqs. (78), (88) and (90) whether the transition to Q_0 may occur or not and, if affirmative, the corresponding switching time t_1 is calculated.

Step 4. (Specifying the optimal discrete sequence). In case the Steps 1–3 yield more than one sequence for the discrete state trajectory, the optimal sequence is determined by comparing the value functions along the obtained discrete state sequences with the value function for the case where no switching or stopping event happens. Finally it should be noted that if Steps 1–3 result in no realized discrete trajectory, then the system may remain in the discrete state Q_0 over the interval $[0, T]$.

5. Simulations

Consider a system of 100 minor agents with two types \mathcal{A}^a and \mathcal{A}^b and a single major agent \mathcal{A}^0 . The minor agents are provided with the option to quit if it is optimal for them to do so. Since the major agent is not permitted to switch in this example, the system has three discrete states Q_0, Q_1, Q_2 , which index the stopping of one or both subpopulations as illustrated in Fig. 2.

To clearly depict the impact of subpopulations' stopping on the control action and the trajectory of other agents, time-varying system matrices for a generic minor agent in subpopulation \mathcal{A}^a , with $N_a = 50$, are defined as

$$A_a \triangleq \begin{bmatrix} 2e^{-t} & e^{-0.5t} \\ e^{-0.5t} & 2e^{-t} \end{bmatrix}, \quad B_a \triangleq \begin{bmatrix} 1 \\ 0.1 \end{bmatrix},$$

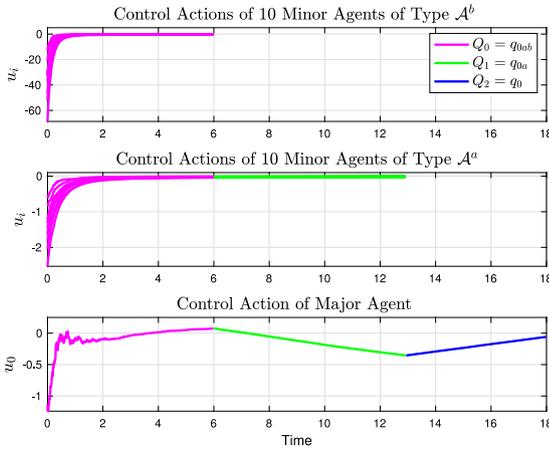


Fig. 3. The control actions for a single realization of the major agent, 10 sample minor agents of type \mathcal{A}^a , and 10 sample minor agents of type \mathcal{A}^b in discrete states Q_0, Q_1, Q_2 .

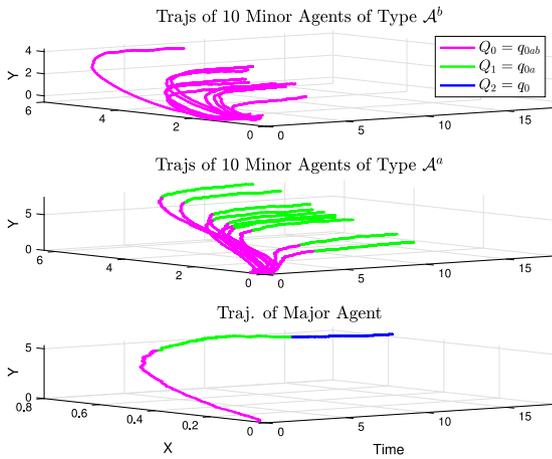


Fig. 4. The state trajectories for a single realization of the major agent, 10 sample minor agents of type \mathcal{A}^a , and 10 sample minor agents of type \mathcal{A}^b in discrete states Q_0, Q_1, Q_2 .

and for a generic minor agent in subpopulation \mathcal{A}^b , with $N_b = 50$, are given by

$$A_b \triangleq \begin{bmatrix} 5e^{-1.5t} \cos(t) & 5e^{-2t} \\ 5e^{-2t} \sin(t) & 5e^{-1.5t} \end{bmatrix}, \quad B_b \triangleq \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

and for the major agent are given by

$$A_0 \triangleq \begin{bmatrix} 2e^{-t} & e^{-t} \\ e^{-0.5t} & 2e^{-0.5t} \end{bmatrix}, \quad B_0 \triangleq \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

The parameters used in the simulation are: $t_{final} = 18$ sec, $\Delta t = 0.01$ sec, $\sigma_0 = 0.015$, $\sigma_a = \sigma_b = 0.05$, $H_0 = 0.6 \times I_2$, $H_1^a = H_1^b = 0.2 \times I_2$, $H_2^a = H_2^b = 0.02 \times I_2$, $G_a = G_b = 0_{2 \times 2}$. We note that [Assumptions 1–4](#) are satisfied as the parameters are bounded analytic functions of time. Then following the steps in [Section 4.5](#) the optimal control actions and state trajectories for a single realization in discrete states Q_0, Q_1, Q_2 for the entire population of 101 agents are obtained. In [Figs. 3 and 4](#) only 10 minor agents are shown for the sake of clarity. The subpopulations \mathcal{A}^b and \mathcal{A}^a stop, respectively, at $t_1 = 6$ sec and $t_2 = 12$ sec. This, in particular, impacts the behaviour of the major agent at the stopping times t_1 and t_2 as illustrated in [Figs. 3 and 4](#).

6. Conclusions

A hybrid mean field game theory has been established here for a class of Major–Minor LQG MFG systems for which controlled switching and stopping times are state and trajectory independent, and only depend on the dynamical and cost functional parameters of each agent. As a result, all agents of the same type would stop or switch at the same time. It is of significant interest to develop and extend the theory to account for switchings and stoppings at individual rates and/or upon arrival on switching manifolds, where individuals in subpopulations may quit or switch to alternative dynamics at different times. This is of particular importance in the modelling of optimal execution problems where traders stop or switch after reaching a specific number of shares. Another future direction is the tractable formulation for several subpopulations, including a systematic methodology for treating more complex discrete state sequence lattices.

Appendix A. Proof of Theorem 1

We invoke the Stochastic Hybrid Minimum Principle ([Pakniyat & Caines, 2016](#)) to form the family of system Hamiltonians as

$$H^{Q_j}(x^{Q_j}, u^{Q_j}, \lambda^{Q_j}, K^{Q_j}) = [K^{Q_j}]^T D^{Q_j} + \frac{1}{2} \left(\|x^{Q_j}(t)\|_{P^{Q_j}(t)}^2 + \|u^{Q_j}(t)\|_{R^{Q_j}(t)}^2 \right) + [\lambda^{Q_j}]^T (A^{Q_j} x^{Q_j} + B^{Q_j} u^{Q_j}), \quad (\text{A.1})$$

It immediately follows that

$$\operatorname{argmin}_{u^{Q_j} \in \mathbb{R}^{m_{Q_j}}} H^{Q_j}(x^{Q_j}, u^{Q_j}, \lambda^{Q_j}, K^{Q_j}) = -[R^{Q_j}]^{-1} [B^{Q_j}]^T \lambda^{Q_j}. \quad (\text{A.2})$$

In order to find conditions under which the adjoint processes take the form $\lambda^{Q_j}(t) = \Pi^{Q_j}(t) x^{Q_j}(t)$ we begin with the final discrete state Q_L , and follow similar arguments as those in classical LQG theory (see e.g. [Yong and Zhou \(1999, Chapter 6\)](#)), to obtain

$$\lambda^{Q_L}(T) = \Pi^{Q_L}(T) x^{Q_L}(T) = \frac{1}{2} \frac{\partial}{\partial x} \|x(T)\|_{P^{Q_L}(T)}^2, \quad (\text{A.3})$$

$$d\lambda^{Q_L} = -\frac{\partial H^{Q_L}}{\partial x}(x^{Q_L}, u^{Q_L}, \lambda^{Q_L}, K^{Q_L}) dt + K^{Q_L} dw \\ = -\left(P^{Q_L} x^{Q_L} + [A^{Q_L}]^T \lambda^{Q_L} \right) dt + K^{Q_L} dw, \quad (\text{A.4})$$

with $K^{Q_L}(t) = \Pi^{Q_L}(t) D^{Q_L}$.

Within the (backward) induction procedure, we hypothesize that $\lambda^{Q_{j+1}}(t) = \Pi^{Q_{j+1}}(t) x^{Q_{j+1}}(t)$ holds and determine conditions under which $\lambda^{Q_j}(t) = \Pi^{Q_j}(t) x^{Q_j}(t)$ is concluded. To this end, we note that from [Pakniyat and Caines \(2016, Theorem 1\)](#) (see also [Pakniyat & Caines, 2017a](#)) the adjoint processes and Hamiltonians must satisfy

$$\lambda^{Q_j}(t_{j+1}) = [\Psi_{\sigma_j}]^T \lambda^{Q_{j+1}}(t_{j+1}+) + C_{\sigma_j} x^{Q_j}(t_{j+1}), \quad (\text{A.5}) \\ H^{Q_j}(x^{Q_j}, u^{Q_j}, \lambda^{Q_j}, K^{Q_j}) - [K^{Q_j}]^T D^{Q_j} - \frac{1}{2} \frac{\partial}{\partial t} \|x^{Q_j}\|_{C_{\sigma_j}^{(t)}}^2 \Big|_{t_{j+1}^-} \\ = H^{Q_{j+1}}(x^{Q_{j+1}}, u^{Q_{j+1}}, \lambda^{Q_{j+1}}, K^{Q_{j+1}}) - [K^{Q_{j+1}}]^T D^{Q_{j+1}} \Big|_{t_{j+1}^-}. \quad (\text{A.6})$$

The substitution of the hypothesized Riccati forms into [\(A.5\)](#) yields

$$\Pi^{Q_j}(t_{j+1}) x^{Q_j}(t_{j+1}-) \\ = [\Psi_{\sigma_j}]^T \Pi^{Q_{j+1}}(t_{j+1}) x^{Q_{j+1}}(t_{j+1}) + C_{\sigma_j} x^{Q_j}(t_{j+1}-) \\ = [\Psi_{\sigma_j}]^T \Pi^{Q_{j+1}}(t_{j+1}) \Psi_{\sigma_j} x^{Q_j}(t_{j+1}-) + C_{\sigma_j} x^{Q_j}(t_{j+1}-) \quad (\text{A.7})$$

and, therefore, if (7) holds, then (A.7) is obtained regardless of the value of $x^{Q_j}(t_{j+1}-)$. Moreover, the substitution of $\lambda^Q(t) = \Pi^Q(t)x^Q(t)$ into (A.2), and subsequently in (A.1), yields

$$H^Q(x, u^o, \lambda, K^Q) - [K^Q]^T D^Q = \frac{1}{2} \|x\|_{P^Q + \Pi^Q A^Q + [A^Q]^T \Pi^Q - \Pi^Q B^Q [R^Q]^{-1} [B^Q]^T \Pi^Q}^2 \quad (\text{A.8})$$

and in particular, at the switching instant t_{j+1} , the substitution of (A.8) and (2) into (A.6) result in

$$\begin{aligned} & \frac{1}{2} [x_{(t_{j+1}-)}^{Q_j}]^T \left(P^{Q_j} + \Pi_{(t_{j+1})}^{Q_j} A^{Q_j} + [A^{Q_j}]^T \Pi_{(t_{j+1})}^{Q_j} \right. \\ & \quad \left. - \Pi_{(t_{j+1})}^{Q_j} B^{Q_j} [R^{Q_j}]^{-1} [B^{Q_j}]^T \Pi_{(t_{j+1})}^{Q_j} - \frac{\partial C_{\sigma_j}(t)}{\partial t} \Big|_{t=t_{j+1}} \right) x_{(t_{j+1}-)}^{Q_j} \\ & = \frac{1}{2} [x_{(t_{j+1})}^{Q_{j+1}}]^T \left(P^{Q_{j+1}} + \Pi_{(t_{j+1})}^{Q_{j+1}} A^{Q_{j+1}} + [A^{Q_{j+1}}]^T \Pi_{(t_{j+1})}^{Q_{j+1}} \right. \\ & \quad \left. - \Pi_{(t_{j+1})}^{Q_{j+1}} B^{Q_{j+1}} [R^{Q_{j+1}}]^{-1} [B^{Q_{j+1}}]^T \Pi_{(t_{j+1})}^{Q_{j+1}} \right) x_{(t_{j+1})}^{Q_{j+1}} \\ & = \frac{1}{2} [\Psi_{\sigma_j} x_{(t_{j+1}-)}^{Q_j}]^T \left(P^{Q_{j+1}} + \Pi_{(t_{j+1})}^{Q_{j+1}} A^{Q_{j+1}} + [A^{Q_{j+1}}]^T \Pi_{(t_{j+1})}^{Q_{j+1}} \right. \\ & \quad \left. - \Pi_{(t_{j+1})}^{Q_{j+1}} B^{Q_{j+1}} [R^{Q_{j+1}}]^{-1} [B^{Q_{j+1}}]^T \Pi_{(t_{j+1})}^{Q_{j+1}} \right) \Psi_{\sigma_j} x_{(t_{j+1}-)}^{Q_j}. \quad (\text{A.9}) \end{aligned}$$

In particular, if (8) holds, then (A.9) holds independent of the realization for $x^{Q_j}(t_{j+1}-)$ and thus, independent of \mathcal{F}_t . Since the satisfaction of (7) and (8) lead to the satisfaction of (A.5) and (A.6) with \mathcal{F}_t -independence, then the induction hypothesis is true due to the uniqueness of the solution to the backward differential Eqs. (A.4). Moreover, (9) and (10) ensure that such a switching instant is unique and therefore the associated Riccati equations and switching conditions globally correspond to a unique optimal strategy.

Appendix B. Jump transition maps and switching costs for minor agents

We define the notation $M_k^{Q_j}(l : m)$, $k \triangleq a, b$, $0 \leq j \leq 3$, which shall be used to identify the switching cost associated with switching time t_j , $1 \leq j \leq 3$, i.e., the cost incurred when a change in the system happens. Matrix $M_k^{Q_j}(l : m)$ is made by making all the entries of $\bar{\mathbb{P}}_k^{Q_j}$ zero except those associated with its l -th to m -th columns and rows, hence it has the same size as $\bar{\mathbb{P}}_k^{Q_j}$, i.e.

$$M_k^{Q_j}(l : m) = \left[\begin{array}{c|c|c} \bar{\mathbb{P}}_k^{Q_j}(\cdot, l : m) & & \\ \hline 0 & & 0 \\ \hline \bar{\mathbb{P}}_k^{Q_j}(l : m, \cdot) & & \\ \hline 0 & & 0 \end{array} \right]_{\text{size}(\bar{\mathbb{P}}_k^{Q_j})} \quad (\text{B.1})$$

where $\bar{\mathbb{P}}_k^{Q_j}(\cdot, l : m)$ and $\bar{\mathbb{P}}_k^{Q_j}(l : m, \cdot)$, respectively, denote l th to m th columns and rows of $\bar{\mathbb{P}}_k^{Q_j}$.

The values of minor agent \mathcal{X}_i^k continuous state before and after the switching time t_j satisfy the following jump transition map

$$x_i^{\text{ex}, Q_j}(t_j) = \Psi_{i, \sigma_j}^k x_i^{\text{ex}, Q_{j-1}}(t_j-), \quad (\text{B.2})$$

where for $k \triangleq a$ (due to space limitation $k \triangleq b$ is skipped)

$$\Psi_{i, \sigma_1}^a = \begin{cases} \Psi_{i, \sigma_{q_1 a}^{q_2 a}} = I_{3n \times 3n}, \\ \Psi_{i, \sigma_{q_1 a}^{q_1 a}} = \begin{bmatrix} I_n & 0_{n \times n} & 0_{n \times n} \\ 0_{n \times n} & I_n & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix}, \\ \Psi_{i, \sigma_{q_1 a}^{q_1 a}} = \begin{bmatrix} I_n & 0_{n \times n} & 0_{n \times n} \\ 0_{n \times n} & I_n & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix}. \end{cases} \quad (\text{B.3})$$

$$\Psi_{i, \sigma_2}^a = \begin{cases} \Psi_{i, \sigma_{q_1 a}^{q_2 a}} = I_{2n \times 2n}, \\ \Psi_{i, \sigma_{q_1 a}^{q_1 a}} = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} \end{bmatrix}, \\ \Psi_{i, \sigma_{q_2 a}^{q_2 a}} = \begin{bmatrix} I_n & 0_{n \times n} & 0_{n \times n} \\ 0_{n \times n} & I_n & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix}, \\ \Psi_{i, \sigma_{q_2 a}^{q_2 b}} = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix}. \end{cases} \quad (\text{B.4})$$

$$\Psi_{i, \sigma_3}^a = \Psi_{i, \sigma_{q_2 a}^{q_2 a}} = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix}. \quad (\text{B.5})$$

The corresponding switching cost weight matrices $\mathbb{C}_{i, \sigma_j}^k$, $k \triangleq a$, are given by

$$\mathbb{C}_{i, \sigma_1}^a = \begin{cases} \mathbb{C}_{i, \sigma_{q_1 a}^{q_2 a}} = 0_{3n \times 3n}, \\ \mathbb{C}_{i, \sigma_{q_1 a}^{q_1 a}} = M_a^{q_1 ab} (3n + 1 : 4n), \\ \mathbb{C}_{i, \sigma_{q_1 a}^{q_1 b}} = \bar{\mathbb{P}}_a^{q_1 ab}, \end{cases} \quad (\text{B.6})$$

$$\mathbb{C}_{i, \sigma_2}^a = \begin{cases} \mathbb{C}_{i, \sigma_{q_1 a}^{q_2 a}} = 0_{2n \times 2n}, \\ \mathbb{C}_{i, \sigma_{q_1 a}^{q_1 a}} = \bar{\mathbb{P}}_a^{q_1 a}, \\ \mathbb{C}_{i, \sigma_{q_2 a}^{q_2 a}} = M_a^{q_2 ab} (3n + 1 : 4n), \\ \mathbb{C}_{i, \sigma_{q_2 a}^{q_2 b}} = \bar{\mathbb{P}}_a^{q_2 ab}, \end{cases} \quad (\text{B.7})$$

$$\mathbb{C}_{i, \sigma_3}^a = \mathbb{C}_{i, \sigma_{q_2 a}^{q_2 a}} = \bar{\mathbb{P}}_a^{q_2 a}. \quad (\text{B.8})$$

Appendix C. Proof of Theorem 3

The ε -Nash property can be shown in two steps for the major agent and a generic minor agent, respectively. Due to space limitation we detail the proof for the major agent and that of a minor agent will follow in the same manner.

(i) Suppose that there exists a sequence $\{\delta_n\}_{n=1}^N$ such that $\delta_N \rightarrow 0$ as $N \rightarrow \infty$, and $\left| \frac{N_k}{N} - \pi_k \right| = o(\delta_N)$, for all $k \triangleq a, b$. Given that all minor agents \mathcal{X}_i^k , $1 \leq i \leq N$, are using the optimal control actions $\mathcal{U}_{MF}^{N_t} \triangleq \{u_i^{Q_j}; 1 \leq i \leq N_t\}$, $1 \leq N_t \leq N < \infty$, given by (85), and the major agent is using an arbitrary control action $u_0 \in \mathcal{U}_g^{N, L}$, we show that

$$(a) \quad \mathbb{E} \|x_0^{Q_j, N} - \bar{x}_0^{Q_j}\|^2 \leq C(o(\frac{1}{N}) + o(\delta_N^2)), \quad (\text{C.1})$$

$$(b) \quad \mathbb{E} \|x^{(N^{Q_j})} - \bar{x}^{Q_j}\|^2 \leq C(o(\frac{1}{N}) + o(\delta_N^2)), \quad (\text{C.2})$$

$$(c) \quad \left| J_0^{Q_j, N}(u_0, u_{-0}^{Q_j, \circ}) - J_0^{Q_j, \infty}(u_0) \right| \leq C(o(\frac{1}{\sqrt{N}}) + o(\delta_N)), \quad (\text{C.3})$$

where $x^{(N^{Q_j})}$, $x_0^{Q_j, N}$ and $\bar{x}_0^{Q_j}$ denote, respectively, the empirical state average, the major agent's state in the finite-population case, and the major agent's state in the infinite-population case, at the discrete state Q_j . Moreover, $J_0^{Q_j, N}(\cdot, \cdot)$ and $J_0^{Q_j, \infty}(\cdot)$ denote, respectively, the major agent's cost functional in the finite-population and the infinite-population cases at the discrete state Q_j . The dynamics governing the major agent in the finite-population and the infinite-population cases for an arbitrary control action $u_0 \in \mathcal{U}_g^{N, L}$ are, respectively, given by

$$dx_0^{Q_j, N} = (A_0^Q x_0^{Q_j, N} + B_0^Q u_0 + F_0^Q x^{(N^{Q_j})}) dt + D_0^Q dw_0,$$

$$dx_0^{Q_j} = (A_0^{Q_j} x_0^{Q_j} + B_0^{Q_j} u_0 + \pi^{Q_j} \otimes F_0^{Q_j} \bar{x}^{Q_j}) dt + D_0^{Q_j} dw_0.$$

Following along the lines of the proofs of Theorem 6 and Proposition 8 in Huang (2010), we can show that (C.1) holds for all $Q_j, 0 \leq j \leq 3$. Furthermore, the difference between the major agent's cost functional in the finite-population and the infinite-population cases is given by

$$\begin{aligned} J_0^{Q_j, N} - J_0^{Q_j, \infty} &= \frac{1}{2} \mathbb{E} \left[\|x_0^{Q_j, N}(T)\|_{P_0^{Q_j}}^2 - \|x_0^{Q_j}(T)\|_{P_0^{Q_j}}^2 \right. \\ &\quad \left. + \int_0^T \left(\|x_0^{Q_j, N}(s) - H_0^{Q_j} x^{(N^{Q_j})}(s)\|_{P_0^{Q_j}}^2 \right. \right. \\ &\quad \left. \left. - \|x_0^{Q_j}(s) - H_0^{Q_j} \bar{x}^{Q_j}(s)\|_{P_0^{Q_j}}^2 \right) ds \right] \\ &= \mathbb{E} \left[\|x_0^{Q_j, N}(T) - x_0^{Q_j}(T)\|_{P_0^{Q_j}}^2 + 2[x_0^{Q_j}(T)]^T \bar{P}_0^{Q_j} \right. \\ &\quad \times (x_0^{Q_j, N}(T) - x_0^{Q_j}(T)) + \int_0^T \left(\|x_0^{Q_j, N}(s) - x_0^{Q_j}(s)\|_{P_0^{Q_j}}^2 \right. \\ &\quad \left. + 2[x_0^{Q_j}(s)]^T P_0^{Q_j} (x_0^{Q_j, N}(s) - x_0^{Q_j}(s)) \right. \\ &\quad \left. + \|x^{(N^{Q_j})}(s) - \bar{x}^{Q_j}(s)\|_{(H_0^{Q_j})^T P_0^{Q_j} H_0^{Q_j}}^2 \right. \\ &\quad \left. + 2[H_0^{Q_j} \bar{x}^{Q_j}(s)]^T P_0^{Q_j} H_0^{Q_j} (x^{(N^{Q_j})}(s) - \bar{x}^{Q_j}(s)) \right. \\ &\quad \left. - 2(x_0^{Q_j, N})^T P_0^{Q_j} H_0^{Q_j} (x^{(N^{Q_j})} - \bar{x}^{Q_j}) \right. \\ &\quad \left. - 2(\bar{x}^{Q_j})^T P_0^{Q_j} H_0^{Q_j} (x_0^{Q_j, N} - x_0^{Q_j}) \right) ds \Big]. \end{aligned} \quad (C.4)$$

Using the Cauchy–Schwarz inequality, we can write

$$\begin{aligned} \mathbb{E} \left[(x_0^{Q_j})^T P_0^{Q_j} (x_0^{Q_j, N} - x_0^{Q_j}) \right] \\ \leq C \mathbb{E} \left[\|x_0^{Q_j}\|^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\|x_0^{Q_j, N} - x_0^{Q_j}\|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (C.5)$$

In a similar manner we can apply the Cauchy–Schwarz inequality to every cross term in (C.4). Hence from (C.1)–(C.2), every cross term is at most of order $(o(\frac{1}{\sqrt{N}}) + o(\delta_N))$, and every squared term is of order $(o(\frac{1}{N}) + o(\delta_N^2))$. Therefore, we obtain (C.3).

(ii) From (C.3) we have

$$\begin{aligned} -o(\delta_N) - o\left(\frac{1}{\sqrt{N}}\right) &\leq J_0^{Q_j, N}(u_0, u_{-0}^{Q_j, \circ}) - J_0^{Q_j, \infty}(u_0) \\ &\leq o(\delta_N) + o\left(\frac{1}{\sqrt{N}}\right). \end{aligned} \quad (C.6)$$

Since (C.6) holds for every u_0 , from its right-hand side

$$J_0^{Q_j, N}(u_0^{Q_j, \circ}, u_{-0}^{Q_j, \circ}) - o(\delta_N) - o\left(\frac{1}{\sqrt{N}}\right) \leq J_0^{Q_j, \infty}(u_0^{Q_j, \circ}), \quad (C.7)$$

and from its left-hand side we have

$$-o(\delta_N) - o\left(\frac{1}{\sqrt{N}}\right) \leq \inf_{u_0} J_0^{Q_j, N}(u_0, u_{-0}^{Q_j, \circ}) - \inf_{u_0} J_0^{Q_j, \infty}(u_0). \quad (C.8)$$

We then isolate $\inf_{u_0} J_0^{Q_j, N}(u_0, u_{-0}^{Q_j, \circ})$ on the right-hand side of (C.8) and use (C.7) to write

$$\begin{aligned} J_0^{Q_j, N}(u_0^{Q_j, \circ}, u_{-0}^{Q_j, \circ}) - 2o(\delta_N) - 2o\left(\frac{1}{\sqrt{N}}\right) \\ \leq \inf_{u_0} J_0^{Q_j, N}(u_0, u_{-0}^{Q_j, \circ}). \quad \square \end{aligned}$$

References

- Aghayeva, Charkaz, & Abushov, Qurban (2011). The maximum principle for the nonlinear stochastic optimal control problem of switching systems. *Global Optimization*, 56(2), 341–352.
- Bensoussan, A., Chau, M. H. M., Lai, Y., & Yam, S. C. P. (2017). Linear-quadratic mean field Stackelberg games with state and control delays. *SIAM Journal on Control and Optimization*, 55(4), 2748–2781.
- Bensoussan, Alain, Frehse, Jens, & Yam, Phillip (2013). *Mean field games and mean field type control theory*. New York: Springer-Verlag.
- Bensoussan, A., & Menaldi, J. L. (1997). Hybrid control and dynamic programming. *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Application and Algorithm*, 3(4), 395–442.
- Bensoussan, A., & Menaldi, J. L. (2000). Stochastic hybrid control. *Mathematical Analysis and Applications*, 249(1), 261–288.
- Bouveret, Géraldine, Dumitrescu, Roxana, & Tankov, Peter (2020). Mean-field games of optimal stopping: A relaxed solution approach. *SIAM Journal on Control and Optimization*, 58(4), 1795–1821.
- Caines, Peter E., & Kizilkale, Arman C. (2017). ε -Nash Equilibria for partially observed LQG mean field games with major player. *IEEE Transactions on Automatic Control*, 62(7), 3225–3234.
- Cardaliaguet, Pierre, Cirant, Marco, & Porretta, Alessio (2018). Remarks on Nash equilibria in mean field game models with a major player. arXiv.
- Cardaliaguet, Pierre, Delarue, François, Lasry, Jean-Michel, & Lions, Pierre-Louis (2019). *The master equation and the convergence problem in mean field games: (AMS-201)*, Vol. 2. Princeton University Press.
- Carmona, René (2020). Applications of mean field games in financial engineering and economic theory. arXiv preprint arXiv:2012.05237.
- Carmona, René, & Delarue, François (2018). *Probabilistic theory of mean field games with applications I-II*. Springer.
- Carmona, René, & Wang, Peiqi (2017). An alternative approach to mean field game with major and minor players, and applications to herders impacts. *Applied Mathematics and Optimization*, 76(1), 5–27.
- Carmona, René, & Zhu, Xiuneng (2016). A probabilistic approach to mean field games with major and minor players. *Annals of Applied Probability*, 26(3), 1535–1580.
- Casgrain, Philippe, & Jaimungal, Sebastian (2020). Mean-field games with differing beliefs for algorithmic trading. *Mathematical Finance*, 30(3), 995–1034.
- Firoozi, Dena (2019). *Mean field games and optimal execution problems: Hybrid and partially observed major minor systems* (Ph.D. thesis), McGill University.
- Firoozi, Dena (2020). LQG mean field games with a major agent: Nash certainty equivalence versus probabilistic approach. arXiv preprint arXiv:2012.04866.
- Firoozi, Dena, & Caines, Peter E. (2017). The execution problem in finance with major and minor traders: A mean field game formulation. In Joseph Apaloo, & Bruno Viscolani (Eds.), *Advances in dynamic and mean field games: Theory, applications, and numerical methods* (pp. 107–130). Cham: Springer International Publishing.
- Firoozi, Dena, & Caines, Peter E. (2019). Belief estimation by agents in major minor LQG mean field games. In *Proceedings of the 58th IEEE conference on decision and control (CDC)* (pp. 1615–1622).
- Firoozi, Dena, & Caines, Peter E. (2021). ε -Nash Equilibria for major minor LQG mean field games with partial observations of all agents. *IEEE Transactions on Automatic Control*, 66(6), 2778–2786.
- Firoozi, Dena, Jaimungal, Sebastian, & Caines, Peter E. (2020). Convex analysis for LQG systems with applications to major–minor LQG mean–field game systems. *Systems & Control Letters*, 142, 104734, (in press).
- Firoozi, Dena, Pakniyat, Ali, & Caines, Peter E. (2017). A mean field game - Hybrid systems approach to optimal execution problems in finance with stopping times. In *Proceedings of the 56th IEEE conference on decision and control (CDC)*. Melbourne, Australia (pp. 3144–3151).
- Garavello, Mauro, & Piccoli, Benedetto (2005). Hybrid necessary principle. *SIAM Journal on Control and Optimization*, 43(5), 1867–1887.
- Gomes, Diogo A., Lafleche, Laurent, & Nurbekyan, Levon (2016). A mean-field game economic growth model. In *2016 American control conference* (pp. 4693–4698).
- Huang, Minyi (2010). Large-population LQG games involving a major player: The Nash certainty equivalence principle. *SIAM Journal on Control and Optimization*, 48(5), 3318–3353.
- Huang, Minyi (2020). Linear-quadratic mean field games with a major player: Nash certainty equivalence versus master equations. *Communications in Information and Systems*, 213–242, Special Issue in Honor of Professor Tyrone Duncan on the Occasion of His 80th Birthday.

- Huang, Minyi, Caines, Peter E., & Malhamé, Roland P. (2003). Individual and mass behavior in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. In *Proceedings of the 42nd IEEE conference on decision and control (CDC)*. Maui, HI (pp. 98–103).
- Huang, Minyi, Caines, Peter E., & Malhamé, Roland P. (2007). Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ε -Nash equilibria. *IEEE Transactions on Automatic Control*, 52(9), 1560–1571.
- Huang, Minyi, Malhamé, Roland P., & Caines, Peter E. (2006). Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information and Systems*, 6(3), 221–252.
- Kordonis, Ioannis, & Papavassilopoulos, George P. (2015). LQ Nash games with random entrance: An infinite horizon major player and minor players of finite horizons. *IEEE Transactions on Automatic Control*, 60(6), 1486–1500.
- Lasry, Jean-Michel, & Lions, Pierre-Louis (2006a). Jeux à champ moyen. i - le cas stationnaire. *Comptes Rendus de l'Académie des Sciences*, 343, 619–625.
- Lasry, Jean-Michel, & Lions, Pierre-Louis (2006b). Jeux à champ moyen. ii - horizon fini et contrôle optimal. *Comptes Rendus de l'Académie des Sciences*, 343, 679–684.
- Lasry, Jean-Michel, & Lions, Pierre-Louis (2007). Mean field games. *Japanese Journal of Mathematics*, 2(1), 229–260.
- Lasry, Jean-Michel, & Lions, Pierre-Louis (2018). Mean-field games with a major player. *Comptes Rendus Mathématique*, 356(8), 886–890.
- Moon, Jun, & Başar, Tamer (2018). Linear quadratic mean field Stackelberg differential games. *Automatica*, 97, 200–213.
- Nguyen, Son Luu, & Huang, Minyi (2012). Linear-quadratic-Gaussian mixed games with continuum-parametrized minor players. *SIAM Journal on Control and Optimization*, 50(5), 2907–2937.
- Nourian, Mojtaba, & Caines, Peter E. (2013). ε -Nash Mean field game theory for nonlinear stochastic dynamical systems with major and minor agents. *SIAM Journal on Control and Optimization*, 51(4), 3302–3331.
- Pakniyat, Ali, & Caines, Peter E. (2016). On the stochastic minimum principle for hybrid systems. In *Proceedings of the 55th IEEE conference on decision and control (CDC)* (pp. 1139–1144).
- Pakniyat, Ali, & Caines, Peter E. (2017a). A class of linear quadratic Gaussian hybrid optimal control problems with realization-independent riccati equations. *IFAC-PapersOnLine*, 50(1), 2241–2246.
- Pakniyat, Ali, & Caines, Peter E. (2017b). Hybrid optimal control of an electric vehicle with a dual-planetary transmission. *Nonlinear Analysis. Hybrid Systems*, 25, 263–282.
- Pakniyat, Ali, & Caines, Peter E. (2017c). On the relation between the minimum principle and dynamic programming for classical and hybrid control systems. *IEEE Transactions on Automatic Control*, 62(9), 4347–4362.
- Pakniyat, Ali, & Caines, Peter E. (2021). On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions. *IEEE Transactions on Automatic Control*, 66(3), 1246–1253.
- Şen, Nevroz, & Caines, Peter E. (2016). Mean field game theory with a partially observed major agent. *SIAM Journal on Control and Optimization*, 54(6), 3174–3224.
- Shaikh, Mohammad S., & Caines, Peter E. (2007). On the hybrid optimal control problem: theory and algorithms. *IEEE Transactions on Automatic Control*, 52(9), 1587–1603, Corrigendum: vol. 54, no. 6, pp. 1428, 2009.
- Shrivats, Arvind, Firoozi, Dena, & Jaimungal, Sebastian (2022a). A mean-field game approach to equilibrium pricing in solar renewable energy certificate markets. *Mathematical Finance*, (in press), available at [arXiv:2003.04938](https://arxiv.org/abs/2003.04938).
- Shrivats, Arvind, Firoozi, Dena, & Jaimungal, Sebastian (2022b). Principal agent mean field games in REC markets. *arXiv preprint arXiv:2112.11963*.

Sussmann, Hector J. (1999). A nonsmooth hybrid maximum principle. In *Lecture notes in control and information sciences*, Vol. 246 (pp. 325–354). London: Springer.

Tarngoo, Farzin, & Caines, Peter E. (2013). On the optimal control of impulsive hybrid systems on Riemannian manifolds. *SIAM Journal on Control and Optimization*, 51(4), 3127–3153.

Yong, Jiongmin, & Zhou, Xun Yu (1999). *Stochastic controls: Hamiltonian systems and HJB equations*. New York: Springer-Verlag.



Dena Firoozi is an Assistant Professor in the Department of Decision Sciences at HEC Montréal (business school of University of Montreal). Before joining HEC Montréal, she was a postdoctoral fellow in the Department of Statistical Sciences at the University of Toronto, Canada, between 2018–2020. She was also a Ph.D. exchange student in the same program during Fall 2017. She completed her Ph.D. in electrical engineering, systems & control, at McGill University, Canada, in 2019. She received her M.Sc. degree from Sharif University of Technology, Iran, in 2011, and her B.Sc. degree from Shiraz University, Iran, in 2009, both in electrical engineering-systems & control. Her research interests include mathematical finance, mean field games, stochastic control and estimation theory.



Ali Pakniyat is an Assistant Professor in the department of Mechanical Engineering at the University of Alabama. He received the B.Sc. degree in Mechanical Engineering from Shiraz University, the M.Sc. degree in Mechanical Engineering from Sharif University of Technology, and the Ph.D. degree in Electrical Engineering from McGill University. After holding a Lecturer position at the Electrical and Computer Engineering department of McGill University, and two postdoctoral positions in the department of Mechanical Engineering at the University of Michigan and the Institute for Robotics and Intelligent Machines at Georgia Tech, he joined the University of Alabama in 2021 where he is now an Assistant Professor in the department of Mechanical Engineering. His research interests include deterministic and stochastic optimal control, nonlinear and hybrid systems, analytical mechanics and chaos, with applications in automotive industry, sensors and actuators, and robotics.



Peter E. Caines received the BA in mathematics from Oxford University in 1967 and the Ph.D. in systems and control theory in 1970 from Imperial College, University of London, supervised by David Q. Mayne, FRS. Following PDF and visiting positions at Stanford, UC Berkeley, Toronto and Harvard, he joined McGill University in 1980, where he is Distinguished James McGill Professor and Macdonald Chair in the Department of Electrical and Computer Engineering. He received the IEEE Control Systems Society Bode Lecture Prize (2009), is a Fellow of IFAC, CIFAR, SIAM, IEEE, the IMA (UK) and the Royal Society of Canada (2003), and is a member of Professional Engineers Ontario. His monograph, *Linear Stochastic Systems* (Wiley, 1988), is now a SIAM Classic and his research interests include stochastic and hybrid systems, and mean field (games and control) systems on complex networks.