Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnline CONFERENCE PAPER ARCHIVE

IFAC-PapersOnLine 48-27 (2015) 169–174

On the Relation between the Hybrid Minimum Principle and Hybrid Dynamic Programming: a Linear Quadratic Example

Ali Pakniyat * **Peter E. Caines** *

* *Centre for Intelligent Machines (CIM) and the Department of Electrical and Computer Engineering (ECE), McGill University*
E-mails: pakniyat@cim.mcgill.ca, peterc@cim.mcgill.ca

Abstract: Hybrid optimal control problems are studied for systems where autonomous and controlled state jumps are allowed at the switching instants and in addition to running costs, switching between discrete states incurs costs. Key aspects of the analysis are the relationship between the Hamiltonian and the adjoint process in the Hybrid Minimum Principle before and after the switching instants, the boundary conditions on the value function in Hybrid Dynamic Programming at these switching times, as well as the relationship between the adjoint process in the Hybrid Minimum Principle and the gradient process of the value function in Hybrid Dynamic Programming. The results are illustrated through an analytic example with linear dynamics and quadratic costs.

Keywords: Optimal Control, Hybrid Systems, Minimum Principle, Dynamic Programming

1. INTRODUCTION

There is now an extensive literature on the optimal control of hybrid systems. On one hand, the generalization of the Pontryagin Maximum Principle (PMP) [Pontryagin et al. (1962)] results in the Hybrid Minimum Principle (HMP) [Clarke and Vinter (1989a,b); Garavello and Piccoli (2005); Passenberg et al. (2011); Shaikh and Caines (2007b); Sussmann (1999a,b); Taringoo and Caines (2011, 2013); Xu and Antsaklis (2004)] that gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system with fixed initial conditions and a sequence of autonomous and controlled switchings. These conditions are expressed in terms of the minimization of the distinct Hamiltonians defined along the hybrid trajectory of the system corresponding to a sequence of discrete states and continuous valued control inputs on the associated time intervals. A feature of special interest in the Hybrid Minimum Principle is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations [Shaikh and Caines (2007a)].

On the other hand, the generalization of Dynamic Programming [Bellman (1966)] for hybrid systems results in the theory of Hybrid Dynamic Programming (HDP) which employs the notion of the optimal cost to go for the hybrid optimal control problem as its fundamental notion. Under the assumption of smoothness of the value function, the Principle of Optimality results in the celebrated Hamilton-Jacobi-Bellman (HJB) equation of HDP [Barles et al. (2010); Bensoussan and Menaldi (1997); Branicky et al. (1998); Caines et al. (2007); Da Silva et al. (2012); Dharmatti and Ramaswamy (2005); Hedlund and Rantzer (2002); Schöllig et al. (2007); Shaikh and Caines (2009)]. In the case of non-smooth value functions, the so-called viscosity solutions give a general class of solutions to the HJB equation [Barles et al. (2010); Bensoussan and Menaldi (1997); Dharmatti and Ramaswamy (2005)].

In past work of the authors (see [Pakniyat and Caines (2013, 2014a,c)]) the results of the Hybrid Minimum Principle are given for the general class of hybrid optimal control problems with autonomous and controlled state jumps and in the presence of a large range of running, terminal and switching costs. It is further proved in [Pakniyat and Caines (2014c)] that the adjoint process in the HMP and the gradient of the value function in HDP are governed by the same dynamic equation and have the same boundary conditions and hence are identical to each other. An illustrative example was provided in [Pakniyat and Caines (2014a)] with scalar nonlinear dynamics equations and nonlinear costs. This paper elaborates the HMP - HDP relationship once more by demonstrating the analytical construction of the value function from its gradient process in the case of hybrid optimal control problems with linear dynamics and quadratic costs.

2. HYBRID OPTIMAL CONTROL PROBLEMS

For the hybrid optimal control problems (HOCP) defined in [Pakniyat and Caines (2013, 2014a,b,c)], consider the infimization of the following hybrid cost

$$J\left(t_0,t_f,h_0,L;I_L\right) := \sum_{i=0}^{L}\int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}(s),u(s)\right)ds + \sum_{j=1}^{L} c_{\sigma_{q_{j-1}q_j}}\left(x_{q_{j-1}}\left(t_j-\right)\right)+g\left(x_{q_L}\left(t_f\right)\right) \quad (1)$$

subject to the dynamics

$$\dot{x}_{q_i}(t) = f_{q_i}\left(x_{q_i}(t),u(t)\right), \;\; a.e.\, t\in[t_i,t_{i+1}), \quad (2)$$

where $0\le i\le L$ and $t_{L+1}=t_f<\infty$, with the initial condition

$$h_0=\left(q_0,x_{q_0}(t_0)\right)=\left(q_0,x_0\right), \quad (3)$$

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.11.170

and the switching jumps maps

$$x_{q_j}(t_j) = \xi\left(x_{q_{j-1}}(t_j-)\right) \equiv \xi\left(\lim_{t\uparrow t_j} x_{q_{j-1}}(t)\right) \tag{4}$$

where $1 \le j \le L$.

□

3. HYBRID MINIMUM PRINCIPLE

Theorem 1 [Pakniyat and Caines (2014a,b,c, 2015)] Define the family of system Hamiltonians by

$$H_{q_j}(x,\lambda,u) = \lambda^T f_{q_j}(x,u) + l_{q_j}(x,u) \tag{5}$$

$x, \lambda \in \mathbb{R}^n, u \in U, q_j \in Q$. Then along the optimal trajectory q^o, x^o, there exists an adjoint process λ^o such that

$$\dot{x}^o = \frac{\partial H_{q^o}}{\partial \lambda}(x^o,\lambda^o,u^o), \tag{6}$$

$$\dot{\lambda}^o = -\frac{\partial H_{q^o}}{\partial x}(x^o,\lambda^o,u^o) \tag{7}$$

almost everywhere $t \in [t_0, t_f]$ with

$$x^o(t_0) = x_0, \tag{8}$$

$$x^o(t_j) = \xi(x^o(t_j-)), \tag{9}$$

$$\lambda^o(t_f) = \nabla g(x^o(t_f)), \tag{10}$$

$$\lambda^o(t_j-) \equiv \lambda^o(t_j) = \nabla\xi^T \lambda^o(t_j+) + p\nabla m + \nabla c_\sigma, \tag{11}$$

where $p \in \mathbb{R}$ when t_j indicates the time of an autonomous switching, and $p = 0$ when t_j indicates the time of a controlled switching.

Moreover, the Hamiltonian is minimized with respect to the control input

$$H_{q^o}(x^o,\lambda^o,u^o) \le H_{q^o}(x^o,\lambda^o,u) \tag{12}$$

for all $u \in U$; and at a switching time t_j the Hamiltonian satisfies

$$H_{q_{j-1}}(t_j-) \equiv H_{q_{j-1}}(t_j) = H_{q_j}(t_j) \equiv H_{q_j}(t_j+) \tag{13}$$

□

4. HYBRID DYNAMIC PROGRAMMING

Consider the hybrid system (2) and the HOCP for the hybrid cost (1). For simplicity of notation, in the rest of the paper and unless otherwise stated, we use x instead of x^o in order to indicate that x refers to the general solution of the corresponding HOCP passing through it. We adapt the same notation for q^o, t_j^o, etc.

At a time $t \in [t_0, t_f]$ that corresponds to some $1 \le j \le L+1$ such that $t \in (t_{j-1}, t_j]$ and for the state $h = (q,x)$ in the hybrid state space, the cost to go function is formed as

$$\begin{aligned} J&\left(t,t_f,q,x,L-j+1;I_{L-j+1}\right) \\ &= \int_t^{t_j} l_q(x,u)\,ds + \sum_{i=j}^{L} c_{\sigma_{q_{i-1}q_i}}\left(t_i, x_{q_{i-1}}(t_i-)\right) \\ &\quad + \sum_{i=j}^{L}\int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds + g\left(x_{q_L}(t_f)\right) \end{aligned} \tag{14}$$

and the value function V is defined as the optimal cost to go over the family of hybrid control inputs, i.e.

$$V(t,q,x,L-j+1) := \inf_{I_{L-j+1}} J\left(t,t_f,q,x,L-j+1;I_{L-j+1}\right) \tag{15}$$

Theorem 2 [Pakniyat and Caines (2014b, 2015)] If at the instant t and the hybrid state (q,x) the value function V is differentiable then it necessarily satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$-\frac{\partial V}{\partial t}(t,q,x,L-j+1) = \inf_u H_q\left(x, \frac{\partial V}{\partial x}(t,q,x,L-j+1),u\right) \tag{16}$$

where

$$H_q\left(x,\frac{\partial V}{\partial x},u\right) := l_q(x,u) + \frac{\partial V}{\partial x}^T f_q(x,u) \tag{17}$$

In addition, the value function satisfies the following terminal time condition

$$V(t_f,q_L,x,0) = g(x) \tag{18}$$

and the boundary conditions

$$H_{q_{j-1}}(t_j-) \equiv H_{q_{j-1}}(t_j) = H_{q_j}(t_j) \equiv H_{q_j}(t_j+) \tag{19}$$

and

$$V(t_j,q,x,L-j+1) = \min_{\sigma\in\Sigma_j}\left\{V(t_j,q_j,\xi_\sigma(x),L-j) + c_\sigma(x)\right\} \tag{20}$$

where $\Sigma_j = \Sigma$ if t_j is a time of a controlled switching. In the case of an autonomous switching, the set Σ_j reduces to a subset of admissible discrete inputs which are consistent with the switching manifold condition $m_{q,q_j}(x) = 0$.

□

5. THE RELATIONSHIP BETWEEN THE MINIMUM PRINCIPLE AND DYNAMIC PROGRAMMING

Theorem 3 [Pakniyat and Caines (2014c)] If the functions f_q and l_q are continuously differentiable for all $q \in Q$, and the value function V is twice continuously differentiable almost everywhere in Lebesgue sense on $\mathbb{R}\times\mathbb{R}^n$ then at all Lebesgue points and times the gradient of the value function and the adjoint process for the corresponding (optimal) switching sequence are governed by the same differential equations, i.e.

$$\frac{d}{dt}\nabla V = -\frac{\partial}{\partial x} f_{q^o}(x^o,u^o)^T \nabla V - \frac{\partial}{\partial x} l_{q^o}(x^o,u^o) \tag{21}$$

and

$$\frac{d}{dt}\lambda^o = -\frac{\partial}{\partial x} f_{q^o}(x^o,u^o)^T \lambda^o - \frac{\partial}{\partial x} l_{q^o}(x^o,u^o) \tag{22}$$

and have the same boundary conditions, i.e.

$$\nabla V\left(t_f,q^o,x(t_f),0\right) = \nabla g\left(x^o(t_f)\right), \tag{23}$$

$$\begin{aligned} \nabla V&\left(t_j-,q_{j-1},x(t_j-),L-j+1\right) \\ &= \nabla\xi|_{x(t_j-)}^T \nabla V(t_j+,q_j,x(t_j+),L-j) \\ &\quad + p\,\nabla m|_{x(t_j-)} + \nabla c|_{x(t_j-)} \end{aligned} \tag{24}$$

for the gradient of the value function, and

$$\lambda^o(t_f) = \nabla g\left(x^o(t_f)\right), \tag{25}$$

$$\lambda^o(t_j-) = \nabla\xi|_{x(t_j-)}^T \lambda^o(t_j+) + p\,\nabla m|_{x(t_j-)} + \nabla c|_{x(t_j-)} \tag{26}$$

for the adjoint process. Hence, from the uniqueness of the solutions of (21) and (22) that are identical almost everywhere on $t \in [t_0,t_f]$, it is concluded that the adjoint process locally describes the gradient of the value function, i.e.

$$\lambda^o = \nabla_x V \tag{27}$$

almost everywhere in Lebesgue sense on $\mathbb{R}\times\mathbb{R}^n$. □

6. ILLUSTRATIVE EXAMPLE

Consider the hybrid system with the indexed vector fields

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f_1(x,u) = \begin{bmatrix} x_2 \\ -x_1 + u \end{bmatrix} \tag{28}$$

and

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f_2(x,u) = \begin{bmatrix} x_2 \\ u \end{bmatrix} \tag{29}$$

where autonomous switchings occur on the switching manifold described by

$$m(x_1(t_s), x_2(t_s)) \equiv x_2(t_s) = 0 \tag{30}$$

Assume that the hybrid optimal control problem is defined as the minimization of the total cost functional

$$J = \int_{t_0}^{t_f} \frac{1}{2}u^2 dt + \frac{1}{2}(x_1(t_s))^2 + \frac{1}{2}\left(x_2\left(t_f\right) - v_{ref}\right)^2 \tag{31}$$

6.1 The HMP Results

Employing the HMP, the corresponding Hamiltonians are defined by

$$H_1 = \lambda_1 x_2 + \lambda_2(-x_1 + u) + \frac{1}{2}u^2 \tag{32}$$

and

$$H_2 = \lambda_1 x_2 + \lambda_2 u + \frac{1}{2}u^2 \tag{33}$$

The Hamiltonian minimization with respect to u (Eq. (12)) gives

$$u^o = -\lambda_2 \tag{34}$$

for both $q = 1$ and $q = 2$.

The adjoint process dynamics (7) is then

$$\dot{\lambda}_1 = \frac{-\partial H_1}{\partial x_1} = \lambda_2 \tag{35}$$

$$\dot{\lambda}_2 = \frac{-\partial H_1}{\partial x_2} = -\lambda_1 \tag{36}$$

for $q = 1$ and

$$\dot{\lambda}_1 = \frac{-\partial H_2}{\partial x_1} = 0 \tag{37}$$

$$\dot{\lambda}_2 = \frac{-\partial H_2}{\partial x_2} = -\lambda_1 \tag{38}$$

for $q = 2$.

The terminal condition for the adjoint process (10) gives

$$\lambda_1\left(t_f\right) = \frac{\partial g}{\partial x_1} = 0 \tag{39}$$

$$\lambda_2\left(t_f\right) = \frac{\partial g}{\partial x_2} = x_2\left(t_f\right) - v_{ref} \tag{40}$$

Eq. (37) together with (39) give

$$\lambda_1 = 0 \qquad t \in \left(t_s, t_f\right] \tag{41}$$

Substituting (41) into (38) gives

$$\dot{\lambda}_2 = 0 \tag{42}$$

and from (40) we get

$$\lambda_2 = x_2\left(t_f\right) - v_{ref} \qquad t \in \left(t_s, t_f\right] \tag{43}$$

The boundary conditions (11) on adjoint processes at the switchings instant give

$$\lambda_1(t_s) = \lambda_1(t_s+) + \frac{\partial c}{\partial x_1} + p\frac{\partial m}{\partial x_1} = x_1(t_s) \tag{44}$$

$$\lambda_2(t_s) = \lambda_2(t_s+) + \frac{\partial c}{\partial x_2} + p\frac{\partial m}{\partial x_2} = x_2\left(t_f\right) - v_{ref} + p \tag{45}$$

Eq. (44) and (45) serve as a terminal condition for the adjoint processes dynamics (35) and (36) which has a general solution of the form

$$\lambda_1 = A\sin(t+\alpha) \qquad t \in [t_0, t_s] \tag{46}$$

$$\lambda_2 = A\cos(t+\alpha) \qquad t \in [t_0, t_s] \tag{47}$$

Then the system state dynamics (6) becomes

$$\dot{x}_1 = \frac{\partial H_1}{\partial \lambda_1} = x_2 \tag{48}$$

$$\dot{x}_2 = \frac{\partial H_1}{\partial \lambda_2} = -x_1 + u^o = -x_1 - \lambda_2 = -x_1 - A\cos(t+\alpha) \tag{49}$$

which has a general solution of the form

$$x_1 = \frac{-1}{2}At\sin(t+\alpha) + B\sin(t+\beta) \tag{50}$$

$$x_2 = \frac{-1}{2}At\cos(t+\alpha) - \frac{1}{2}A\sin(t+\alpha) + B\cos(t+\beta) \tag{51}$$

that need to satisfy the initial condition $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$. At the switching time t_s the continuity condition on x_1 and x_2 is deduced from (9)

$$x_1(t_s+) \equiv x_1(t_s) = x_1(t_s-) \tag{52}$$

$$x_2(t_s+) \equiv x_2(t_s) = x_2(t_s-) = 0 \tag{53}$$

which form the initial conditions required for

$$\dot{x}_1 = \frac{\partial H_2}{\partial \lambda_1} = x_2 \tag{54}$$

$$\dot{x}_2 = \frac{\partial H_2}{\partial \lambda_2} = u^o = -\lambda_2 = v_{ref} - x_2\left(t_f\right) \tag{55}$$

The solution of these equations is given as

$$x_1 = x_1(t_s) + \frac{1}{2}\left(v_{ref} - x_2\left(t_f\right)\right)(t-t_s)^2 \tag{56}$$

$$x_2 = \left(v_{ref} - x_2\left(t_f\right)\right)(t-t_s) \tag{57}$$

The equation (57) is expressed in terms of $x_2\left(t_f\right)$. In order to write an explicit form for x_2 we evaluate (57) at t_f to write

$$x_2\left(t_f\right) = \left(v_{ref} - x_2\left(t_f\right)\right)\left(t_f - t_s\right) \tag{58}$$

or

$$x_2\left(t_f\right)\left(1 + t_f - t_s\right) = v_{ref}\left(t_f - t_s\right) \tag{59}$$

Thus

$$x_2\left(t_f\right) = \frac{v_{ref}\left(t_f - t_s\right)}{1 + t_f - t_s} \tag{60}$$

Substituting (60) into (56) and (57) we get

$$x_1 = x_1(t_s) + \frac{v_{ref}}{2\left(1 + t_f - t_s\right)}(t-t_s)^2 \tag{61}$$

$$x_2 = \frac{v_{ref}}{1 + t_f - t_s}(t-t_s) \tag{62}$$

for $t \in \left[t_s, t_f\right]$. The Hamiltonian continuty condition at t_s gives

$$\lambda_1(t_s-)x_2(t_s-) - \lambda_2(t_s-)(x_1(t_s-) + \lambda_2(t_s-)) + \frac{1}{2}\lambda_2(t_s-)^2$$
$$= \lambda_1(t_s+)x_2(t_s+) - \lambda_2(t_s+)^2 + \frac{1}{2}\lambda_2(t_s+)^2 \tag{63}$$

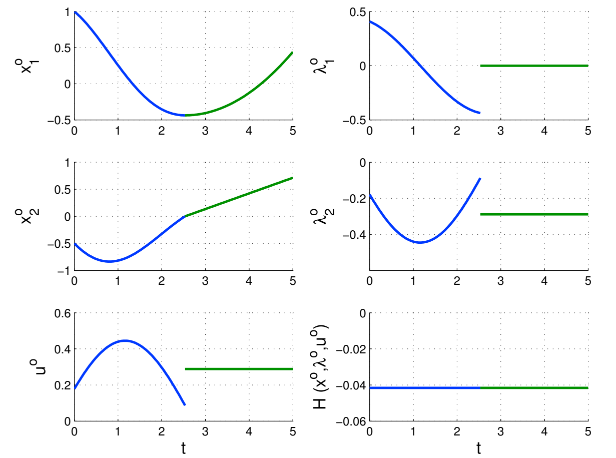

Fig. 1. The optimal trajectory components x_1^o and x_2^o, the corresponding adjoint process components λ_1^o and λ_2^o, the optimal control input u^o and the corresponding Hamiltonian $H\left(x^o,\lambda^o,u^o\right)$ for $t_0=0$, $x_{10}=1$, $x_{20}=-0.5$, $t_f=5$ and $v_{ref}=1$

With a change of signs on both sides of the equation and using the equations (52) and (53) this equality becomes

$$\lambda_2\left(t_s-\right)x_{1s}+\frac{1}{2}\lambda_2\left(t_s-\right)^2=\frac{1}{2}\lambda_2\left(t_s+\right)^2 \quad (64)$$

Hence, by solving simultaneously the following 6 equations for the given $t_0=0$, $x\left(t_0\right)\equiv\left[x_{10},x_{20}\right]^T$, t_f and v_{ref} the values of the 6 unkown parameters A,α,B,β,t_s and p are determined.

$$B\sin\beta=x_{10} \quad (65)$$

$$-\frac{1}{2}A\sin\left(\alpha\right)+B\cos\left(\beta\right)=x_{20} \quad (66)$$

$$A\left(1+\frac{t_s}{2}\right)\sin\left(t_s+\alpha\right)=B\sin\left(t_s+\beta\right) \quad (67)$$

$$A\cos\left(t_s+\alpha\right)=\frac{v_{ref}}{1+t_f-t_s}+p \quad (68)$$

$$\frac{-1}{2}At_s\cos\left(t_s+\alpha\right)-\frac{1}{2}A\sin\left(t_s+\alpha\right)+B\cos\left(t_s+\beta\right)=0 \quad (69)$$

$$A\cos\left(t_s+\alpha\right)\left(\frac{-1}{2}At_s\sin\left(t_s+\alpha\right)+B\sin\left(t_s+\beta\right)\right)+\frac{1}{2}A^2\cos^2\left(t_s+\alpha\right)=\frac{1}{2}\left(\frac{v_{ref}}{1+t_f-t_s}\right)^2 \quad (70)$$

where (65) and (66) are derived from the substitution of the system initial condition in (50) and (51); equations (67) and (68) are derived from the adjoint boundary conditions (44) and (45); the relation (69) is the result of the switching manifold condition (53), and (70) represents the Hamiltonian continuity condition (64). For the values of $t_0=0$, $x_{10}=1$, $x_{20}=-0.5$, $t_f=5$ and $v_{ref}=1$ the results are demonstrated in Fig. 1.

6.2 HDP results from their relationship to the HMP results

Employing Theorem 3 we want to construct the value function satisfying the results of Theorem 2. To this end we rewrite the system dynamics equations (28) and (29) and the ∇V dynamics equation (21) in the matrix form, i.e. for q_1 the first dynamics equations become

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\\nabla_{x_1}V\\\nabla_{x_2}V\end{bmatrix}=\begin{bmatrix}0&1&0&0\\-1&0&0&-1\\0&0&0&1\\0&0&-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\nabla_{x_1}V\\\nabla_{x_2}V\end{bmatrix} \quad (71)$$

and for q_2 the second dynamics equations are presented as

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\\nabla_{x_1}V\\\nabla_{x_2}V\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&0&-1\\0&0&0&0\\0&0&-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\nabla_{x_1}V\\\nabla_{x_2}V\end{bmatrix} \quad (72)$$

The state transition matrix for these equations are denoted by ϕ_i, where

$$\phi_1\left(t,t_0\right)=\begin{bmatrix}\cos\delta&\sin\delta&\frac{-\delta\sin\delta}{2}&\frac{-\delta\cos\delta}{2}\\-\sin\delta&\cos\delta&\frac{-\sin\delta-\delta\cos\delta}{2}&\frac{-\cos\delta+\delta\sin\delta}{2}\\0&0&\cos\delta&\sin\delta\\0&0&-\sin\delta&\cos\delta\end{bmatrix} \quad (73)$$

with $\delta:=t-t_0$, and

$$\phi_2\left(t,t_s\right)=\begin{bmatrix}1&\left(t-t_s\right)&\frac{\left(t-t_s\right)^3}{6}&\frac{-\left(t-t_s\right)^2}{2}\\0&1&\frac{\left(t-t_s\right)^2}{2}&-\left(t-t_s\right)\\0&0&1&0\\0&0&-\left(t-t_s\right)&1\end{bmatrix} \quad (74)$$

Partitioning ϕ_2, the solution of (72) for $t\in\left(t_s,t_f\right]$ can be written as

$$x\left(t_f\right)=\phi_{2,11}\left(t_f,t\right)x\left(t\right)+\phi_{2,12}\left(t_f,t\right)\nabla V\left(t\right) \quad (75)$$

$$\nabla V\left(t_f\right)=\phi_{2,21}\left(t_f,t\right)x\left(t\right)+\phi_{2,22}\left(t_f,t\right)\lambda\left(t\right) \quad (76)$$

which gives

$$\begin{bmatrix}x_1\left(t_f\right)\\x_2\left(t_f\right)\end{bmatrix}=\begin{bmatrix}1&\left(t_f-t\right)\\0&1\end{bmatrix}\begin{bmatrix}x_1\left(t\right)\\x_2\left(t\right)\end{bmatrix}+\begin{bmatrix}\frac{\left(t_f-t\right)^3}{6}&\frac{-\left(t_f-t\right)^2}{2}\\\frac{\left(t_f-t\right)^2}{2}&-\left(t_f-t\right)\end{bmatrix}\begin{bmatrix}\nabla_{x_1}V\left(t\right)\\\nabla_{x_2}V\left(t\right)\end{bmatrix} \quad (77)$$

and

$$\begin{bmatrix}\nabla_{x_1}V\left(t_f\right)\\\nabla_{x_2}V\left(t_f\right)\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\left(t\right)\\x_2\left(t\right)\end{bmatrix}+\begin{bmatrix}1&0\\-\left(t_f-t\right)&1\end{bmatrix}\begin{bmatrix}\nabla_{x_1}V\left(t\right)\\\nabla_{x_2}V\left(t\right)\end{bmatrix} \quad (78)$$

Substituting the above expressions for $x\left(t_f\right)$ and $\nabla V\left(t_f\right)$ into (23) results in

$$\begin{bmatrix}\nabla_{x_1}V\left(t_f\right)\\\nabla_{x_2}V\left(t_f\right)\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}\left(\begin{bmatrix}x_1\left(t_f\right)\\x_2\left(t_f\right)\end{bmatrix}-\begin{bmatrix}0\\v_{ref}\end{bmatrix}\right)=\begin{bmatrix}0\\x_2\left(t_f\right)-v_{ref}\end{bmatrix} \quad (79)$$

which from (78) gives

$$\begin{bmatrix}\nabla_{x_1}V\left(t\right)\\\nabla_{x_2}V\left(t\right)\end{bmatrix}=\begin{bmatrix}0\\x_2^o\left(t_f\right)-v_{ref}\end{bmatrix} \qquad t\in\left(t_s,t_f\right] \quad (80)$$

Substitution of $x_2\left(t_f\right)$ from (77) gives

$$\nabla V\left(t\right)=\begin{bmatrix}0&0\\0&\frac{1}{t_f-t+1}\end{bmatrix}\left(x\left(t\right)-\begin{bmatrix}0\\v_{ref}\end{bmatrix}\right) \quad (81)$$

Thus the value function in the second dynamics with no switching ahead (i.e. after the switching time t_s) should be of the form

$$V(t,q_2,x,0)=\frac{1}{2}x^T\begin{bmatrix}0 & 0\\ 0 & \dfrac{1}{t_f-t+1}\end{bmatrix}x+\begin{bmatrix}0 & \dfrac{-v_{ref}}{t_f-t+1}\end{bmatrix}x+\alpha_2(t) \tag{82}$$

Since V in (82) satisfies the HJB equation (16), the time dependant constant term $\alpha_2(t)$ is determined to be

$$\alpha_2(t)=\frac{v_{ref}^2}{2\left(t_f-t+1\right)} \tag{83}$$

giving the value function in the second dynamics with no remaining switching as

$$V(t,q_2,x,0)=\frac{1}{2}x^T\begin{bmatrix}0 & 0\\ 0 & \dfrac{1}{t_f-t+1}\end{bmatrix}x+\begin{bmatrix}0 & \dfrac{-v_{ref}}{t_f-t+1}\end{bmatrix}x+\frac{v_{ref}^2}{2\left(t_f-t+1\right)}=\frac{\left(v_{ref}-x_2\right)^2}{2\left(t_f-t+1\right)} \tag{84}$$

Simalrly, partitioning ϕ_1 in (73) which is the solution of (71) in $t\in[t_0,t_s)$ gives

$$x(t_s-)=\phi_{1,11}(t_s,t)\,x(t)+\phi_{1,12}(t_s,t)\,\nabla V(t) \tag{85}$$

$$\nabla V(t_s)=\phi_{1,21}(t_s,t)\,x(t)+\phi_{1,22}(t_s,t)\,\nabla V(t) \tag{86}$$

which is equivalent to

$$\begin{bmatrix}x_1(t_s)\\ x_2(t_s)\end{bmatrix}=\begin{bmatrix}\cos(t_s-t) & \sin(t_s-t)\\ -\sin(t_s-t) & \cos(t_s-t)\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}\dfrac{-(t_s-t)\sin(t_s-t)}{2} & \dfrac{-(t_s-t)\cos(t_s-t)}{2}\\ \dfrac{-\sin(t_s-t)-(t_s-t)\cos(t_s-t)}{2} & \dfrac{-\cos(t_s-t)+(t_s-t)\sin(t_s-t)}{2}\end{bmatrix}\begin{bmatrix}\nabla_{x_1}V(t)\\ \nabla_{x_2}V(t)\end{bmatrix}, \tag{87}$$

$$\begin{bmatrix}\nabla_{x_1}V(t_s)\\ \nabla_{x_2}V(t_s)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}\cos(t_s-t) & \sin(t_s-t)\\ -\sin(t_s-t) & \cos(t_s-t)\end{bmatrix}\begin{bmatrix}\nabla_{x_1}V(t)\\ \nabla_{x_2}V(t)\end{bmatrix} \tag{88}$$

From (24) $\nabla V(t_s)$ is given as

$$\begin{bmatrix}\nabla_{x_1}V(t_s)\\ \nabla_{x_2}V(t_s)\end{bmatrix}=\begin{bmatrix}\nabla_{x_1}V(t_s+)\\ \nabla_{x_2}V(t_s+)\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t_s)\\ x_2(t_s)\end{bmatrix}+p\begin{bmatrix}0\\ 1\end{bmatrix} \tag{89}$$

where $\nabla V(t_s+)$ is determined from (81). Hence,

$$\begin{bmatrix}\nabla_{x_1}V(t_s)\\ \nabla_{x_1}V(t_s)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & \dfrac{1}{t_f-t_s+1}\end{bmatrix}\begin{bmatrix}x_1(t_s)\\ x_2(t_s)\end{bmatrix}+\begin{bmatrix}0\\ p-\dfrac{v_{ref}}{t_f-t_s+1}\end{bmatrix} \tag{90}$$

Substituting $x^o(t_s)$ and $\lambda^o(t_s)$ from (87) and (88) respectively results in

$$\begin{bmatrix}\cos(t_s-t) & \sin(t_s-t)\\ -\sin(t_s-t) & \cos(t_s-t)\end{bmatrix}\begin{bmatrix}\nabla_{x_1}V(t)\\ \nabla_{x_2}V(t)\end{bmatrix}=\begin{bmatrix}\cos(t_s-t) & \sin(t_s-t)\\ \dfrac{-\sin(t_s-t)}{t_f-t_s+1} & \dfrac{\cos(t_s-t)}{t_f-t_s+1}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ p-\dfrac{v_{ref}}{t_f-t_s+1}\end{bmatrix}$$
$$+\begin{bmatrix}\dfrac{-(t_s-t)\sin(t_s-t)}{2} & \dfrac{-(t_s-t)\cos(t_s-t)}{2}\\ \dfrac{-\sin(t_s-t)-(t_s-t)\cos(t_s-t)}{2\left(t_f-t_s+1\right)} & \dfrac{-\cos(t_s-t)+(t_s-t)\sin(t_s-t)}{2\left(t_f-t_s+1\right)}\end{bmatrix}\begin{bmatrix}\nabla_{x_1}V(t)\\ \nabla_{x_1}V(t)\end{bmatrix} \tag{91}$$

which can be expressed as

$$\begin{bmatrix}\nabla_{x_1}V(t)\\ \nabla_{x_2}V(t)\end{bmatrix}=\begin{bmatrix}\cos\delta_s+\dfrac{\delta_s\sin\delta_s}{2} & \sin\delta_s+\dfrac{\delta_s\cos\delta_s}{2}\\ \dfrac{\delta_s\cos\delta_s-\left(2\delta_{fs}+1\right)\sin\delta_s}{2\left(\delta_{fs}+1\right)} & \dfrac{\left(2\delta_{fs}+3\right)\cos\delta_s-\delta_s\sin\delta_s}{2\left(\delta_{fs}+1\right)}\end{bmatrix}^{-1}\left(\begin{bmatrix}\cos\delta_s & \sin\delta_s\\ \dfrac{-\sin\delta_s}{\delta_{fs}+1} & \dfrac{\cos\delta_s}{\delta_{fs}+1}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ p-\dfrac{v_{ref}}{\delta_{fs}+1}\end{bmatrix}\right) \tag{92}$$

with

$$\delta_s:=t_s-t,\quad \delta_{fs}:=t_f-t_s \tag{93}$$

Hence,

$$\nabla V(t,q_1,x,1)=\begin{bmatrix}K_{1,11}(t) & K_{1,12}(t)\\ K_{1,21}(t) & K_{1,22}(t)\end{bmatrix}x+\begin{bmatrix}s_{1,1}(t)\\ s_{1,2}(t)\end{bmatrix} \tag{94}$$

where

$$\begin{aligned}K_{1,11}(t)&=\frac{2+\left(\delta_{fs}+\frac{1}{2}\right)\left(1+\cos 2\delta_s\right)}{2\left(\delta_{fs}+1\right)-\frac{\delta_s^2}{2}+\cos 2\delta_s+\delta_{fs}\delta_s\sin 2\delta_s}\\ K_{1,12}(t)&=\frac{\left(\delta_{fs}+\frac{1}{2}\right)\sin 2\delta_s-\delta_s}{2\left(\delta_{fs}+1\right)-\frac{\delta_s^2}{2}+\cos 2\delta_s+\delta_{fs}\delta_s\sin 2\delta_s}\\ K_{1,21}(t)&=\frac{\left(\delta_{fs}-\frac{1}{2}\right)\sin 2\delta_s-\delta_s}{2\left(\delta_{fs}+1\right)-\frac{\delta_s^2}{2}+\cos 2\delta_s+\delta_{fs}\delta_s\sin 2\delta_s}\\ K_{1,22}(t)&=\frac{2+\left(\delta_{fs}-\frac{1}{2}\right)\left(1-\cos 2\delta_s\right)}{2\left(\delta_{fs}+1\right)-\frac{\delta_s^2}{2}+\cos 2\delta_s+\delta_{fs}\delta_s\sin 2\delta_s}\end{aligned} \tag{95}$$

and

$$\begin{aligned}s_{1,1}(t)&=\frac{\left(\delta_{fs}+1\right)\left(p-\frac{v_{ref}}{\delta_{fs}+1}\right)\left(-2\sin\delta_s-\delta_s\cos\delta_s\right)}{2\left(\delta_{fs}+1\right)-\frac{\delta_s^2}{2}+\cos 2\delta_s+\delta_{fs}\delta_s\sin 2\delta_s}\\ s_{1,2}(t)&=\frac{\left(\delta_{fs}+1\right)\left(p-\frac{v_{ref}}{\delta_{fs}+1}\right)\left(2\cos\delta_s+\delta_s\sin\delta_s\right)}{2\left(\delta_{fs}+1\right)-\frac{\delta_s^2}{2}+\cos 2\delta_s+\delta_{fs}\delta_s\sin 2\delta_s}\end{aligned} \tag{96}$$

with δ_s and δ_{fs} defined in (93). The values of t_s and p are determined from the switching manifold condition (30), i.e.

$$x_2(t_s)=\begin{bmatrix}x_1(t) & x_2(t)\end{bmatrix}\begin{bmatrix}K_{1,21}(t)-\sin(t_s-t)\\ K_{1,22}(t)+\cos(t_s-t)\end{bmatrix}+s_{1,2}(t)=0 \tag{97}$$

as well as the Hamiltonian boundary condition (19) which gives

$$p-\frac{v_{ref}}{t_f-t_s+1}=-\frac{x_1^o(t_s)}{2}\pm\frac{1}{2}\sqrt{\left(\frac{2v_{ref}}{t_f-t_s+2}\right)^2+\left(x_1^o(t_s)\right)^2} \tag{98}$$

i.e. at any instant t and for any given continuous state $x(t)$ the value function for the first dynamics with one switching left is locally described by

$$V(t,q_1,x,1)=\frac{1}{2}x^TK_1(t)\,x+s_1(t)^T x+\alpha_1(t) \tag{99}$$

with

$$K_1(t)=\begin{bmatrix}K_{1,11}(t) & K_{1,12}(t)\\ K_{1,21}(t) & K_{1,22}(t)\end{bmatrix}, \tag{100}$$

$$s_1(t)=\begin{bmatrix}s_{1,1}(t)\\ s_{1,2}(t)\end{bmatrix} \tag{101}$$

and $\alpha_1(t)$ determined from the substitution of (99) in the HJB equation (16) which gives its dynamics as

$$\dot{\alpha}_1 = \frac{1}{2}\left(s_{1,2}(t)\right)^2 \quad (102)$$

and its value at t_s determined from the value function boundary condition (20) as

$$\alpha_1(t_s) = \alpha_2(t_s) = \frac{v_{ref}^2}{2\left(t_f - t_s + 1\right)} \quad (103)$$

In summary, the solution to the HJB partial differential equation of Hybrid Dynamic Programming (16) is determined indirectly from its gradient process ∇V which is uniquely identified by the HMP - HDP relationship equations presented in Theorem 3. The HJB equation is employed only for the determination of the time-varying constant terms $\alpha_i(t)$ which do not appear in ∇V the gradient of the value function. □

ACKNOWLEDGEMENT

This work is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Automotive Partnership Canada (APC).

REFERENCES

Barles, G., Dharmatti, S., Ramaswamy, M., 2010. Unbounded Viscosity Solutions of Hybrid Control Systems. ESAIM - Control, Optimisation and Calculus of Variations 16 (1), 176–193.

Bellman, R., 1966. Dynamic Programming. Science 153 (3731), 34–37.

Bensoussan, A., Menaldi, J. L., 1997. Hybrid Control and Dynamic Programming. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Application and Algorithm 3 (4), 395–442.

Branicky, M. S., Borkar, V. S., Mitter, S. K., 1998. A Unified Framework for Hybrid Control: Model and Optimal Control Theory. IEEE Transactions on Automatic Control 43 (1), 31–45.

Caines, P. E., Egerstedt, M., Malhame, R., Schöllig, A., 2007. A Hybrid Bellman Equation for Bimodal Systems. In: Proceedings of the 10th International Conference on Hybrid Systems: Computation and Control, Pisa, Italy. Vol. 4416 LNCS. pp. 656–659.

Clarke, F. H., Vinter, R. B., 1989a. Applications of Optimal Multiprocesses. SIAM Journal on Control and Optimization 27 (5), 1048–1071.

Clarke, F. H., Vinter, R. B., 1989b. Optimal Multiprocesses. SIAM Journal on Control and Optimization 27 (5), 1072–1091.

Da Silva, J. E., De Sousa, J. B., Pereira, F. L., 2012. Dynamic Programming Based Feedback Control for Systems with Switching Costs. In: Proceedings of the IEEE International Conference on Control Applications, Dubrovnik, Croatia. pp. 634–639.

Dharmatti, S., Ramaswamy, M., 2005. Hybrid Control Systems and Viscosity Solutions. SIAM Journal on Control and Optimization 44 (4), 1259–1288.

Garavello, M., Piccoli, B., 2005. Hybrid Necessary Principle. SIAM Journal on Control and Optimization 43 (5), 1867–1887.

Hedlund, S., Rantzer, A., 2002. Convex Dynamic Programming for Hybrid Systems. IEEE Transactions on Automatic Control 47 (9), 1536–1540.

Pakniyat, A., Caines, P. E., 2013. The Hybrid Minimum Principle in the Presence of Switching Costs. In: Proceedings of the 52nd IEEE Conference on Decision and Control, Florence, Italy. pp. 3831–3836.

Pakniyat, A., Caines, P. E., 2014a. On the Minimum Principle and Dynamic Programming for Hybrid Systems. In: Proceedings of the 19th International Federation of Automatic Control World Congress IFAC, Cape Town, South Africa. pp. 9629–9634.

Pakniyat, A., Caines, P. E., 2014b. On the Minimum Principle and Dynamic Programming for Hybrid Systems. Research Report, Department of Electrical and Computer Engineering (ECE), McGill University, July 2014.

Pakniyat, A., Caines, P. E., 2014c. On the Relation between the Minimum Principle and Dynamic Programming for Hybrid Systems. In: Proceedings of the 53rd IEEE Conference on Decision and Control, Los Angeles, CA, USA. pp. 19–24.

Pakniyat, A., Caines, P. E., 2015. On the Minimum Principle and Dynamic Programming for Hybrid Systems with Low Dimensional Switching Manifolds. In: Proceedings of the 54th IEEE Conference on Decision and Control, Osaka, Japan.

Passenberg, B., Leibold, M., Stursberg, O., Buss, M., 2011. The Minimum Principle for Time-Varying Hybrid Systems with State Switching and Jumps. In: Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, USA. pp. 6723–6729.

Pontryagin, L., Boltyanskii, V., Gamkrelidze, R., Mishchenko, E., 1962. The Mathematical Theory of Optimal Processes. Vol. 4. Wiley Interscience, New York.

Schöllig, A., Caines, P. E., Egerstedt, M., Malhamé, R., 2007. A hybrid Bellman Equation for Systems with Regional Dynamics. In: Proceedings of the 46th IEEE Conference on Decision and Control, New Orleans, LA, USA. pp. 3393–3398.

Shaikh, M. S., Caines, P. E., 2007a. On Relationships between Weierstrass-Erdmannn Corner Condition, Snell's Law and the Hybrid Minimum Principle. In: Proceedings of International Bhurban Conference on Applied Sciences and Technology, IBCAST. pp. 117–122.

Shaikh, M. S., Caines, P. E., 2007b. On the Hybrid Optimal Control Problem: Theory and Algorithms. IEEE Transactions on Automatic Control 52 (9), 1587–1603, Corrigendum: vol. 54, no. 6, 2009, p 1428.

Shaikh, M. S., Caines, P. E., 2009. A Verification Theorem for Hybrid Optimal Control Problem. In: Proceedings of the IEEE 13th International Multitopic Conference, INMIC. pp. 1–3.

Sussmann, H. J., 1999a. A Nonsmooth Hybrid Maximum Principle. Lecture Notes in Control and Information Sciences, Springer London, Volume 246, 325–354.

Sussmann, H. J., 1999b. Maximum Principle for Hybrid Optimal Control Problems. In: Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, AZ, USA. pp. 425–430.

Taringoo, F., Caines, P., 2011. Gradient Geodesic and Newton Geodesic HMP Algorithms for the Optimization of Hybrid Systems. Annual Reviews in Control 35 (2), 187–198.

Taringoo, F., Caines, P. E., 2013. On the Optimal Control of Impulsive Hybrid Systems on Riemannian Manifolds. SIAM Journal on Control and Optimization 51 (4), 3127–3153.

Xu, X., Antsaklis, P. J., 2004. Optimal Control of Switched Systems based on Parameterization of the Switching Instants. IEEE Transactions on Automatic Control 49 (1), 2–16.