Time Optimal Hybrid Minimum Principle and the Gear Changing Problem for Electric Vehicles

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Abstract: The statement of the Hybrid Minimum Principle is presented for time optimal control problems where autonomous and controlled state jumps at the switching instants are accompanied by changes in the dimension of the state space. A key aspect of the analysis is the relationship between the Hamiltonians and the adjoint processes before and after the switching instants. As an example application, an electric vehicle equipped with a two-speed seamless transmission, that augments an additional degree of freedom during the transition period, is modelled within this framework. The state-dependent motor torque constraints are converted to state-independent control input constraints via a change of variable and the introduction of auxiliary discrete states. The problem of the minimum acceleration time required for reaching the speed of 100 km/hr is formulated within the presented framework and the Time Optimal Hybrid Minimum Principle is employed in order to find the optimal control inputs and the optimal gear changing instants.

Keywords: Optimal Control, Hybrid Systems, Minimum Principle, Electric Vehicle, Gear Selection

1. INTRODUCTION

There is now an extensive literature on the optimal control of hybrid systems. With the exception of the studies on Hybrid Dynamic Programming (see e.g. [Bensoussan and Menaldi (1997); Branicky et al. (1998); Dharmatti and Ramaswamy (2005); Pakniyat and Caines (2014a,c, 2015); Schöllig et al. (2007)]) and the reachability approach for specific applications (e.g. [Lygeros, Tomlin, and Sastry (1997)]) the majority of research on the optimal control of hybrid systems is focused on the Hybrid Minimum Principle (HMP) [Clarke and Vinter (1989); Garavello and Piccoli (2005); Pakniyat and Caines (2013, 2014a,c, 2015); Passenberg et al. (2011); Shaikh and Caines (2007); Sussmann (1999); Taringoo and Caines (2013); Xu and Antsaklis (2004)] which is the generalization of the fundamental Pontryagin Maximum Principle. The HMP gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system with fixed initial conditions and a sequence of autonomous and controlled switchings. These conditions are expressed in terms of the minimization of the distinct Hamiltonians defined along the hybrid trajectory of the system corresponding to a sequence of discrete states and continuous valued control inputs on the associated time intervals. A feature of special interest is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations.

In past work of the authors (see [Pakniyat and Caines (2013, 2014a,c)]) the statement of the Hybrid Minimum Principle is presented for the general class of hybrid optimal control problems with autonomous and controlled state jumps and in the presence of a large range of running, terminal and switching costs. The aim of this paper is to formulate the gear changing problem for electric vehicles in the hybrid optimal control framework. To this end, we extend the formulation presented in [Pakniyat and Caines (2014d)] for gear-equipped electric vehicles with the inclusion of the transmission dynamics by considering the model of a seamless dual break transmission system reported in [Rahimi M., Pakniyat, and Boulet (2014)]. Due to the special structure of the transmission system considered, the mechanical degree of freedom and hence, the dimension of the (continuous) state space of the system are dependant on the status of the transmission, i.e. whether a gear number is fixed or the system is undergoing a transition between the two gears. In order to avoid state-dependant input constraints, the torque constraints for the electric motor which have a special type of speed dependence (see Fig. 1) are converted to state-independent control input constraints with a change of variables and the introduction of auxiliary discrete states. With the association of discrete states to these physical and auxiliary statuses of the transmission and the electric motor, a hybrid model is derived that satisfies the basic assumptions required for the statement of the Time Optimal Hybrid Minimum Principle. Employing this theorem, the problem of the minimum acceleration time required for reaching the speed of 100 km/hr (or 60 mph) from the stationary state is solved and the results are demonstrated.

2. BASIC ASSUMPTIONS

A hybrid system (structure) is defined as a septuple
\[ \mathcal{H} = \{ H := Q \times M, I := \Sigma \times U, \Gamma, A, F, \Xi, \mathcal{M} \} \]
where the symbols in the expression are defined as below.

A0: \( Q = \{ 1, 2, \ldots, |Q| \} \equiv \{ q_1, q_2, \ldots, q_{|Q|} \}, |Q| < \infty \), is a finite set of discrete states (components).

\( M = \{ \mathbb{R}^{n_q} \}_{q \in Q} \) is a family of finite dimensional continuous state spaces, where \( n_q \leq n < \infty \) for all \( q \in Q \).

A1: \( \Sigma = \{ \sigma_1, \sigma_2, \ldots, \sigma_s \} \) is a finite set of discrete states (components).

A2: For each \( q \in Q \), there is a unique \( \sigma_q \) \in \( \Sigma \).

A3: \( \Gamma = \{ \tau_{i,j} \} \equiv \{ \tau_{q_1q_2} \} \) is a finite set of switching times.

A4: \( A = \{ a_{i,j} \} \equiv \{ a_{q_1q_2} \} \) is a finite set of autonomous state transition functions.

A5: \( F = \{ f_{i,j} \} \equiv \{ f_{q_1q_2} \} \) is a finite set of controlled state transition functions.

A6: \( \Xi = \{ \xi_{i,j} \} \equiv \{ \xi_{q_1q_2} \} \) is a finite set of state dependent switching functions.

A7: \( \mathcal{M} = \{ \mathcal{M}_i \} \equiv \{ \mathcal{M}_{q_1} \} \) is a family of control inputs.

A8: \( \mathcal{M}_i \) is a family of control inputs.

A9: \( \mathcal{M}_i \) is a family of control inputs.

A10: \( \mathcal{M}_i \) is a family of control inputs.

A11: \( \mathcal{M}_i \) is a family of control inputs.
\( H := Q \times M \) is called the (hybrid) state space of the hybrid system \( \mathbb{H} \).

\( I := \Sigma \times U \) is the set of system input values, where \( |\Sigma| < \infty \) and \( U = \{ U_q \}_q \subset \mathbb{R}^n \) is the set of admissible input control values, where \( U_q \) is a compact set in \( \mathbb{R}^n \).

The set of admissible (continuous) control inputs \( \mathcal{W}(U) := L_{\infty}([t_0, T], U) \), is defined to be the set of all measurable functions that are bounded up to a set of measure zero on \([t_0, T], T \leq \infty\). The boundedness properly necessarily holds since admissible input functions take values in the compact set \( U \).

\( \Gamma : \mathbb{H} \times \Sigma \rightarrow \mathbb{H} \) is a time independent (partially defined) discrete state transition map.

\( \Xi : \mathbb{H} \times \Sigma \rightarrow \mathbb{H} \) is a time independent (partially defined) continuous state jump transition map.

\( A : Q \times \Sigma \rightarrow Q \) denotes both a finite automaton and the automaton’s associated transition function on the state space \( Q \) and event set \( \Sigma \), such that for any discrete state \( q \in Q \) only the discrete controlled and uncontrolled transitions into the \( q \)-dependant subset \( A(q, \sigma) \subset Q \) occur under the projection of \( \Gamma \) on its \( Q \) components: \( \Gamma : Q \times \mathbb{R}^n \times \Sigma \rightarrow H_Q \).

\( F \) is an indexed collection of vector fields \( \{ f_q \}_q \subset Q \) such that \( f_q \in C^\infty(\mathbb{R}^n \times U_q \to \mathbb{R}^n) \), \( k_{f_q} \geq 1 \), satisfies a uniform (in \( x \)) Lipschitz condition, i.e. there exists \( L_f < \infty \) such that \( \| f_q(x, u) - f_q(x, u) \| \leq L_f \| x_{1} - x_{2} \| \), \( x_{1}, x_{2} \in \mathbb{R}^n \), \( u \in U_q \), \( q \in Q \).

\( \mathcal{E} \) denotes a collection of switching manifolds such that, for any ordered pair \( \alpha = (p, q) \), \( m_{\alpha} \) is a smooth, i.e. \( C^\infty \), codimension 1 sub-manifold of \( \mathbb{R}^n \), described locally by \( m_{\alpha} = \{ x : m_{\alpha}(x) = 0 \} \).

**AI:** The initial state \( h_0 := (q_0, x(0)) \in H \) is such that \( m_{q_0, q_0}(x_0) \neq 0 \), for all \( q_j \in Q \).

### 3. TIME OPTIMAL HYBRID MINIMUM PRINCIPLE

Consider the initial time \( t_0 \), initial hybrid state \( h_0 := (q_0, x_0) \) and the terminal hybrid state \( h_f := (q_f, x_f) \) to be reached in a finite time \( t_f < \infty \). Let

\[
S_L = \left\{ (t_0, i_d), (t_1, \sigma_{q_0}), \ldots, (t_{L-1}, \sigma_{q_{L-2}}, q_{L-1}), (t_{L}, \sigma_{q_{L-1}}, q_{L}) \right\}
\]

\[
\equiv \left\{ (t_0, q_0), (t_1, q_1), \ldots, (t_{L-1}, q_{L-1}), (t_L, q_L) \right\}
\]

be a hybrid switching sequence and let \( L_t := (S_L, u) \in \mathcal{E} \) be a hybrid input trajectory which subject to A0 and A1 results in a (necessarily unique) hybrid state process (see [Shaikh and Caines (2007)]) and is such that \( L \) controlled and autonomous switchings occur on the time interval \([t_0, T], L \leq t_f \). In this paper, the number of switchings \( L_t \) is held fixed and we denote the corresponding set of inputs by \( \{ I_L \} \).

Define the hybrid cost as

\[
J(t_0, t_f, h_0, L; I_L) := t_f \equiv \sum_{i=0}^{L-1} \int_{t_i}^{t_{i+1}} dt
\]

subject to

\[
x_q(t_i) = f_q(x_q(t_i), u(t_i)), \quad a.e. \; t \in [t_i, t_{i+1}),
\]

\[
x_q(t_0) = x_0,
\]

\[
x_q(t_f) = x_f
\]

where \( 0 \leq i < L, 1 \leq j < L \) and \( t_{L+1} = t_f \).

Then the Hybrid Optimal Control Problem (HOC) is to find the infimum \( J^* \) over the family of input trajectories \( \{ I_L \} \), i.e.

\[
J^*(t_0, t_f, h_0, L) = \inf \{ J(t_0, t_f, h_0, L; I_L) \}
\]

**Theorem 1** [Pakniyat and Caines (2014b)] Consider the hybrid system \( \mathbb{H} \) together with the assumptions A0 and A1 as above and the HOC (8) for the hybrid cost (3). Define the family of system Hamiltonians by

\[
H_{q_j}(x, \lambda, u) = \lambda^T f_{q_j}(x, u) + 1
\]

\[
x, \lambda \in \mathbb{R}^n, u \in U_q, q_j \in Q.\]

Then for an optimal switching sequence \( q^* \) and along the corresponding optimal trajectory \( x^* \), there exists an adjoint process \( \lambda^* \) such that

\[
x^* = \frac{\partial H_{q_j}}{\partial \lambda}(x^*, \lambda^*, u^*)
\]

\[
\lambda^* = -\frac{\partial H_{q_j}}{\partial x}(x^*, \lambda^*, u^*)
\]

almost everywhere \( t \in [t_0, t_f] \) with

\[
x^*(t_0) = x_0,
\]

\[
x^*(t_f) = \xi(x^*(t_{f-})),
\]

\[
x^*(t_{f-}) = x_{f-}.
\]

\[
\lambda^*(t_{f-}) = \lambda^*(t_f) = \nabla \xi^T \lambda^*(t_{f+}) + p \nabla m
\]

where \( p \in \mathbb{R} \) when \( t_f \) indicates the time of an autonomous switching, and \( p = 0 \) when \( t_f \) indicates the time of a controlled switching. Moreover, the Hamiltonian is minimized with respect to the control input, i.e.

\[
H_{q_j}(x^*, \lambda^*, u^*) \leq H_{q_j}(x^*, \lambda^*, u^*)
\]

for all \( u \in U_{q_j} \); at the terminal time \( t_f \) the Hamiltonian equals zero

\[
H_{q_j}(t_f) = 0
\]

and at a switching time \( t_f \) the Hamiltonian satisfies

\[
H_{q_{j-1}}(t_f) = H_{q_{j}}(t_{f+}) + \frac{\partial m}{\partial t}
\]

### 4. ELECTRIC VEHICLE WITH TRANSMISSION

#### 4.1 Vehicle Dynamics

According to Newton’s second law of motion, with \( m \) being the effective mass of the vehicle and \( z \) the coordinate along the road, the car’s acceleration \( a = dv/dt \) depends on the resultant of the traction force \( F_t \), the aerodynamic force \( 1/2 \rho C_d A v^2 \), the gravitational force along the road \( mg \sin \gamma(z) \) and the rolling resistance force \( mgC_r \cos \gamma(z) \).

Thus the system dynamics is described by

\[
\frac{dz}{dt} = v
\]

\[
\frac{dv}{dt} = -\frac{1}{m} F_t - \frac{1}{2m} \rho C_d A v^2 - g \sin \gamma(z) - g C_r \cos \gamma(z)
\]
Assuming that the road has zero grading (i.e. $\gamma(z) \equiv 0$) the dynamics for the car speed $v$ becomes decoupled from its position $z$. The traction force $F_t$ is related to the motor torque $T_m$ through the transmission system consisting of a dual planetary gear set (presented in the following section) and the differential with the gear ratios $G_{R_i}$ and $i_{fd}$ respectively where the index $i$ in $G_{R_i}$ represents the gear number. At each gear number $i$ the dynamics of the car is described as

$$\frac{dv}{dt} = -\frac{D_0C_dA_f}{2m}v^2 + \frac{i_{fd}GR_i}{mR_w}T_m - gC_r$$ (20)

### 4.2 Dynamics of the Transmission

The full derivation of the dynamics of the dual planetary transmission system considered in this paper is reported in detail in [Rahimi M., Pakniyat, and Boulet (2014)]. In summary, the two-stage planetary gear sets provide a constant gear ratio when either the common sun gears or the common ring gears are held fixed. At these states, the input and output speeds and torques are geometrically related by the gear ratios of the transmission. In the first gear, the ring gear of the transmission is held fixed i.e. $\omega_r = 0$ resulting in the gear ratio

$$GR_1 = \frac{1 + R_2}{1 + R_1}$$ (21)

and in the second gear, the sun gear is held fixed i.e. $\omega_S = 0$ resulting in the gear ratio

$$GR_2 = \frac{(1 + R_2)R_1}{(1 + R_1)R_2}$$ (22)

During the gear changing, however, the mechanical degree of freedom is increased by one and the number of control inputs is increased by two. Namely, the minimum number of states required to present the dynamics of the car during the gear shifting process is two (selected here to be $\omega_S$ and $\omega_r$ the angular velocity of the sun gear and the ring gear of the transmission) and, in addition to the motor torque $T_m$, the brake torque acting on the sun gear $T_{BR}$ and the brake torque acting on the ring gear $T_{BB}$ influence the dynamics during the shifting period.

Considering that the input speed, which is the angular velocity of the motor $\omega_M$, and the output speed, which is the car velocity $v$, are geometrically related to the considered states according to the equations

$$\omega_M = \frac{\omega_S + R_1\omega_R}{1 + R_1}$$ (23)

and

$$v = \frac{R_w(\omega_S + R_2\omega_R)}{i_{fd}(1 + R_2)}$$ (24)

the dynamics system equations of the vehicle is derived from the transmission system dynamics in [Rahimi M., Pakniyat, and Boulet (2014); Eq. (13)] as

$$\dot{\omega}_S = -A_{S\omega}\omega_S + A_{SR}\omega_R - A_{SA}(\omega_S + R_2\omega_R)^2 + B_{SSTBS} - B_{SRTB} + B_{SMTM} - D_{S}}$$ (25)

$$\dot{\omega}_R = A_{RS}\omega_S - A_{RR}\omega_R - A_{RA}(\omega_S + R_2\omega_R)^2 - B_{RSTBS} - B_{RRTB} + B_{RMTM} - D_{RL}$$ (26)

with $T_{BS} \in [-|T_{BS}|^{max},0]$ and $T_{BR} \in [-|T_{BR}|^{max},0]$ and where the coefficients are introduced in Appendix A.

### 4.3 Electric Motor

The electric motor considered in this paper has specifications similar to the TM4 MOTIVE A® motor whose torque is constrained as a function of its speed according to Figure 1, namely

$$|T_M| \leq T_{max}^\text{limit}$$ (27)

and

$$|T_M\omega_M| \leq P_{max}^\text{limit}$$ (28)

with $T_{max}^\text{limit} = 200N.m$ and $P_{max}^\text{limit} = 80kW$.

In order to avoid mixed state and input constraints like (28) we define a change of variable by the introduction of

$$u = \frac{T_M}{T_{max}^\text{limit}}, \quad \omega_M < \omega^*$$ (29)

$$u = \frac{T_M\omega_M}{P_{max}^\text{limit}}, \quad \omega_M \geq \omega^*$$ (30)

with $\omega^* = 400\text{rad/sec}$. Thus the constraints (27) and (28) will both become $u \in [-1,1]$ which lies within the assumption A0 requiring $U$ to be an invariant compact set.

### 4.4 Hybrid System Formulation

In order to present the system dynamics in the hybrid framework presented in section 2, the following discrete states are assigned to each continuous dynamics of the system: $q_1$ with $x = [\nu] \in \mathbb{R}$ corresponds to the dynamics in the first gear and in the maximum torque limit region with the vector field

$$\dot{x} = f_1(x,u) = -A_x\nu^2 - B_1u - C_1g$$ (31)

where

$$A_x = \frac{\rho_dC_dA_f}{2m}$$ (32)

and

$$B_1 = \frac{i_{fd}GR_1}{mR_w}T_{max}^\text{limit}$$ (33)

When the motor speed $\omega_M = \frac{i_{fd}GR_1u}{R_w}$ reaches $\omega^* = 400\text{rad/sec}$ the system autonomously switches to $q_2$ with $x = [\nu] \in \mathbb{R}$ which corresponds to the dynamics in the first gear and in the maximum power limit region possessing the vector field

$$\dot{x} = f_2(x,u) = -A_x\nu^2 - B_2\frac{u}{x} - C_2g$$ (34)

with

$$B_2 = \frac{P_{max}^\text{limit}}{m}$$ (35)

The switching manifold $m_{q_1q_2}$ is thus represented as

$$m_{q_1q_2}(x) \equiv x - \frac{\omega^*R_w}{i_{fd}GR_1} = 0$$ (36)
Due to space limitation and according to the manoeuvre studied in this paper, the dynamics for the maximum torque limit region during the gear changing process and in the second gear are eliminated and thus we assign $q_3$ with $x = [\omega_8, \omega_9]^T \in \mathbb{R}^2$ to the dynamics in the maximum power limit region during the gear changing with the vector field
\[
\dot{x} = f_3(x, u)
\]  
(37)

where
\[
\dot{x}_1 = f_3^{(1)}(x, u) = -A_{SS}x_1 + A_{SR}x_2 - A_{SA}(x_1 + R_2x_2)^2 + B_{SS}T_{BS} - B_{SR}T_{BR} + B_{SM}P_{M}^{\text{max}}(1 + R_1) \frac{u}{x_1 + R_1x_2} - D_{SL},
\]
\[
\dot{x}_2 = f_3^{(2)}(x, u) = A_{RS}x_1 - A_{RR}x_2 - A_{RA}(x_1 + R_2x_2)^2 - B_{RS}T_{BS} - B_{RR}T_{BR} + B_{RM}P_{M}^{\text{max}}(1 + R_1) \frac{u}{x_1 + R_1x_2} - D_{RL}.
\]  
(38)

The jump map corresponding to the (controlled) transition between $q_2$ and $q_3$ is described by
\[
x(t_{s_2}) = \xi_{q_2 \rightarrow q_3}(x(t_{s_2}^-)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t_{s_2}^-)
\]  
(39)

where the jump map $\xi_{q_2 \rightarrow q_3} : \mathbb{R} \rightarrow \mathbb{R}^2$ from $x(t_{s_2}^-) \in \mathbb{R}$ to $x(t_{s_2}) \in \mathbb{R}^2$ is differentiable.

When the speed of the sun gear $\omega_8$ becomes zero the system switches to $q_4$ with $x = [\nu]^T \in \mathbb{R}$ that corresponds to the dynamics in the second gear and in the maximum power limit region and the vector field becomes
\[
\dot{x} = f_4(x, u) = -A_o \nu^2 - B_4 \frac{u}{\nu} + C_i \nu
\]  
with
\[
B_4 = \frac{P_{M}^{\text{max}}}{m}
\]  
(41)

Note that although $q_2$ and $q_4$ has the same dynamics equation in terms of the normalized control input $u$, the motor torque $T_M$ and its speed $\omega_M$ are different in these two dynamics. The switching manifold corresponding to the transition from $q_3$ to $q_4$ is described as
\[
m_{q_3 \rightarrow q_4}(x) \equiv x_4 = 0
\]  
(42)

and the jump map corresponding to this transition is described by
\[
x(t_{s_3}) = \xi_{q_3 \rightarrow q_4}(x(t_{s_3}^-)) = \begin{bmatrix} R_w \\ i/d \end{bmatrix} \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} x(t_{s_3}^-)
\]  
(43)

with $\xi_{q_3 \rightarrow q_4} : \mathbb{R}^2 \rightarrow \mathbb{R}$ differentiable.

5. TIME OPTIMAL ACCELERATION

The hybrid optimal control problem considered in this paper is the minimization of the acceleration period required for reaching the top speed of $100 \frac{km}{h} = 27.8 \frac{m}{s} \approx 60 \text{mph}$ starting from the stationary state in the first gear and terminating in the second gear. Hence, the cost to be minimized is
\[
J(u, T_{BS}, T_{BR}; t_{s_1}, t_{s_2}, t_{s_3}) = \int_{t_{s_1}}^{t_{s_2}} dt + \int_{t_{s_2}}^{t_{s_3}} dt + \int_{t_{s_3}}^{t_f} dt + \int_{t_f}^{t_{s_1}} dt
\]  
(44)

with $t_f$ being the first time that $x(t) = 27.8 \frac{m}{s}$ is satisfied.

The family of system Hamiltonians are formed as
\[
H_1(x, \lambda, u) = 1 + \lambda \left(-A_o x^2 - B_1 u - C_i \nu\right)
\]  
(45)
\[
H_2(x, \lambda, u) = 1 + \lambda \left(-A_o x^2 - B_2 \frac{u}{\nu} + C_i \nu\right)
\]  
(46)
\[
H_3(x, \lambda, u, T_{BS}, T_{BR}) = 1 + \lambda_1 \left(-A_{SS}x_1 + A_{SR}x_2 - A_{SA}(x_1 + R_2x_2)^2 + B_{SS}T_{BS} - B_{SR}T_{BR} + B_{SM}P_{M}^{\text{max}}(1 + R_1) \frac{u}{x_1 + R_1x_2} - D_{SL}\right) + \lambda_2 \left(A_{RS}x_1 - A_{RR}x_2 - A_{RA}(x_1 + R_2x_2)^2 - B_{RS}T_{BS} - B_{RR}T_{BR} + B_{RM}P_{M}^{\text{max}}(1 + R_1) \frac{u}{x_1 + R_1x_2} - D_{RL}\right)
\]  
(47)

and
\[
H_4(x, \lambda, u, \nu) = 1 + \lambda \left(-A_o \nu^2 - B_4 \frac{u}{\nu} - C_i \nu\right)
\]  
(48)

Then according to the Time Optimal Hybrid Minimum Principle in section 3, the adjoint process dynamics is determined as
\[
\dot{\lambda}_1 = -\frac{\partial H_1}{\partial x} = (2A_o)x \lambda, \quad t \in [t_0, t_{s_1}]
\]  
(49)
\[
\dot{\lambda}_2 = -\frac{\partial H_2}{\partial x} = \begin{bmatrix} -2A_o & B_1 \end{bmatrix} \lambda, \quad t \in (t_{s_1}, t_{s_2})
\]  
(50)
\[
\dot{\lambda}_3 = -\frac{\partial H_3}{\partial x} = \begin{bmatrix} -A_{SS} & -A_{SR} \end{bmatrix} \lambda, \quad t \in (t_{s_2}, t_{s_3})
\]  
(51)

with
\[
\lambda_1 = -\frac{\partial H_1}{\partial \lambda} =
\]
\[
\lambda_2 = -\frac{\partial H_2}{\partial \lambda} =
\]
\[
\lambda_3 = -\frac{\partial H_3}{\partial \lambda} =
\]

The boundary conditions for $\lambda$ are determined from Eq. (15) as
\[
\lambda(t_f) = \nabla \xi_{q_3 \rightarrow q_4}^T \lambda(t_{s_2}) + p_3 \nabla m_{q_3 \rightarrow q_4}
\]  
(52)
\[
= \begin{bmatrix} R_w \\ i/d \end{bmatrix} \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} \lambda(t_{s_2}) + p_3
\]  
(55)

\[
\lambda(t_{s_3}) = \nabla \xi_{q_3 \rightarrow q_4}^T \lambda(t_{s_2}) + p_3
\]  
(56)

\[
\lambda(t_{s_4}) = \lambda(t_{s_1}) + p_1
\]  
(57)

It can be easily verified that for the above dynamics and boundary conditions, the adjoint process has a negative sign for all $t \in [t_0, t_f]$ (see also Fig. 2) and hence the Hamiltonian minimization condition (16) results in $u^*(t) = 1$ for $t \in [t_0, t_f]$ as well as $T_{BS}^*(t) = -|T_{BS}|_{\text{max}}$ and $T_{BR}^*(t) = 0$ for $t \in [t_{s_2}, t_{s_3}]$. 

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Fig. 2. The car speed, the adjoint processes and the corresponding Hamiltonians for the minimum acceleration period problem

The Hamiltonian terminal condition (17) gives

\[
H_4 (x(t_f), \lambda (t_f), u(t_f)) = 1 + \lambda (t_f) \left(-A, x(t_f) - R u(t_f) - C, g\right) = 0 \tag{58}
\]

and the Hamiltonian continuity at switching instants is deduced from (18) as

\[
H_3 (x, \lambda, u)(t_i- -) = H_4 (x, \lambda, u)(t_i+), \quad i = 1, 2 \tag{59}
\]

\[
H_2 (x, \lambda, u)(t_3-) = H_3 (x, \lambda, u)(t_3+), \quad t_3 = 2.901 \tag{60}
\]

\[
H_1 (x, \lambda, u)(t_0-) = H_2 (x, \lambda, u)(t_0+), \quad t_0 = 0.444 \tag{61}
\]

The results for the parameter values presented in Appendix B are illustrated in Figure 2. For better illustration, the speed of the vehicle is shown in km/hr and, in addition, the components \(\lambda_1\) and \(\lambda_2\) of the adjoint process in \(t \in [t_2, t_3]\) are multiplied by \(i_{rd}(1 + R) / R_c\) and \(i_{rd}(1 + R) / (R_c R_2)\) respectively so that the boundary conditions (55) and (56) can be verified more easily. The optimal values for the switching and final times are \(t_{1\ast} = 0.444, t_{2\ast} = 2.901, t_{3\ast} = 4.014, t_f = 6.042\).

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REFERENCES


Appendix A. TRANSMISSION DYNAMICS DERIVATION

The dynamics equation for the transmission is presented in [Rahimi M., Pakniyat, and Boulet (2014); Eq. (13)] as

\[
\dot{\omega}_S = \frac{1}{a} \left( -C_S \tau \omega_S + C_R \lambda \dot{\omega}_R + \tau [T_{BS} + T_{SF}] - \lambda [T_{BR} + T_{RF}] + \varepsilon T_M - d T_I \right) \tag{A.1}
\]

\[
\dot{\omega}_R = \frac{1}{a} \left( C_S \lambda \omega_S - C_R \gamma \omega_R - \lambda [T_{BS} + T_{SF}] + \gamma [T_{BR} + T_{RF}] + \varepsilon T_M - f T_I \right) \tag{A.2}
\]

With this transmission mounted on a vehicle with the dynamics equation (20) the load torque \( T_I \) on the transmission is related to the resistance forces on the car and the acceleration term in the form of

\[
T_I = R_u \left( \frac{pC_d A_f R_w^2 (\omega_S + R_w \omega_R)^2}{2 \tau f_d (1 + R_w)^2} + C_{mg} \frac{m R_w (\omega_S + R_w \omega_R)}{i_{fd} (1 + R_w)} \right) \tag{A.3}
\]

Substituting (A.3) into (A.1) and (A.2) and solving for the explicit equations for \( \dot{\omega}_S \) and \( \dot{\omega}_R \), the equations (25) and (26) are achieved with the parameters related to the values presented in [Rahimi M., Pakniyat, and Boulet (2014)] and in Table B.2 by

\[
A_{BS} = \frac{C_S \left( a f_d \left( 1 + R_w \right) \tau + \left( f \tau + d \lambda \right) R_2 m R_w^2 \right)}{a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2}
\]

\[
A_{BR} = \frac{C_R \left( a f_d \left( 1 + R_w \right) \lambda + \left( f \lambda + d \gamma \right) R_2 m R_w^2 \right)}{a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2}
\]

\[
A_{SA} = \frac{2 i_{fd} \left( 1 + R_w \right) \left( a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2 \right)}{a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2}
\]

\[
B_{BS} = \frac{\tau \left( 1 + R_w \right) + \lambda \left( f \lambda + d \gamma \right) R_2 m R_w^2}{a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2}
\]

\[
B_{BR} = \frac{\tau \left( 1 + R_w \right) + \lambda \left( f \lambda + d \gamma \right) R_2 m R_w^2}{a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2}
\]

\[
B_{SM} = \frac{2 i_{fd} \left( 1 + R_w \right) \left( a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2 \right)}{a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2}
\]

\[
D_{SL} = \frac{\tau \left( 1 + R_w \right) + \lambda \left( f \lambda + d \gamma \right) R_2 m R_w^2}{a f_d \left( 1 + R_w \right) + a \left( d + R_2 \right) f m R_w^2}
\]

\[
\text{Table B.2. The parameters considered for the transmission}
\]

<table>
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<th>Unit</th>
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<td>kg</td>
</tr>
<tr>
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<td>m</td>
</tr>
<tr>
<td>( \rho )</td>
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<td>m( ^3 )</td>
</tr>
<tr>
<td>( A_f )</td>
<td>2</td>
<td>m( ^2 )</td>
</tr>
<tr>
<td>( C_d )</td>
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<td>–</td>
</tr>
<tr>
<td>( C_f )</td>
<td>0.02</td>
<td>–</td>
</tr>
<tr>
<td>( \tau )</td>
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<td>m( ^{-1} )</td>
</tr>
<tr>
<td>( i_{fd} )</td>
<td>12</td>
<td>–</td>
</tr>
</tbody>
</table>

Appendix B. PARAMETER VALUES

The car parameters considered in this paper are presented in Table B.1 and the values for the parameters of the transmission are brought from [Rahimi M., Pakniyat, and Boulet (2014)] and are presented in Table B.2.