The Quest for Missing Component: Dualities in Hybrid Optimal Control

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Dynamics and Control Systems Laboratory (DCSL) Meeting

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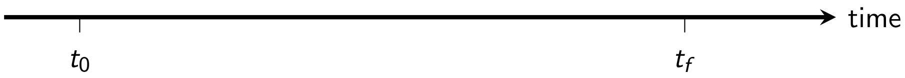
Control Policy Determination



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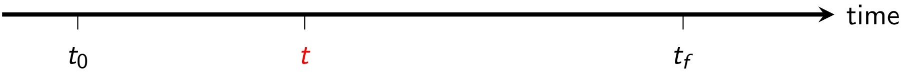
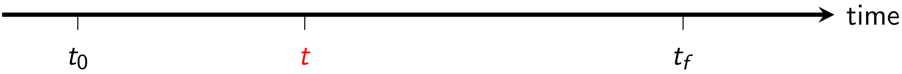


- Information at time t: Everything that has happened in $[t_0, t]$
- Prediction at time t: Everything that might happen in $(t, t_f]$

Control Policy Determination



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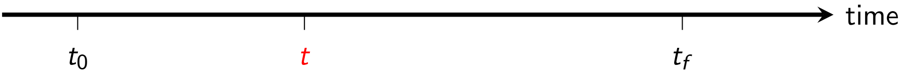

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- Closed-Loop Policy: $\text{Input}_t = \text{Function}(\text{t}, \text{Information}_t, \text{Prediction}_t)$

Control Policy Determination

- **Information** at time t: Everything that has happened in $[t_0, t]$
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- Closed-Loop Policy: $\text{Input}_t = \text{Function}(\text{t}, \text{Information}_t, \text{Prediction}_t)$
- Feedback Policy: $\text{Input}_t = \text{Function}(\text{t}, \text{State}_t, \boxed{(?)}_t)$

Insufficiency of Pure State Feedback

Stochastic Hybrid Dynamics

$$dx_t = \left(A_{q_0} x_t + B_{q_0} u_t\right) dt + G\, dw_t$$

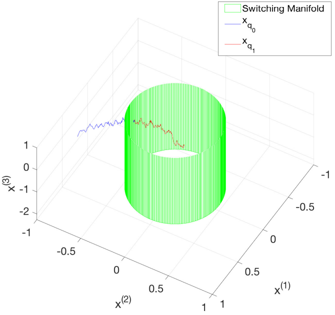
$$dx_t = \left(A_{q_1} x_t + B_{q_1} u_t\right) dt + G\, dw_t$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Switching Manifold

$$\begin{bmatrix} x^{(1)}_{\tau-} & x^{(2)}_{\tau-} & x^{(3)}_{\tau-} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^{(1)}_{\tau-} \\ x^{(2)}_{\tau-} \\ x^{(3)}_{\tau-} \end{bmatrix} = r^2$$

$$x(\tau-) \equiv x_{\tau-} := \lim_{t \uparrow \tau} x_t$$



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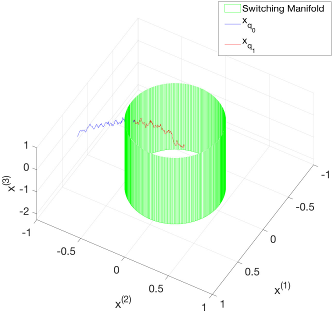

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Optimal Input Structure

$$u_t^o = \text{Function}\left(t,\ x_t, \underset{\Im^t}{\mathbb{E}}^{\boldsymbol{u}}\{\tau\}, \underset{\Im^t}{\mathbb{E}}^{\boldsymbol{u}}\{x_{\tau-}\}\right)$$

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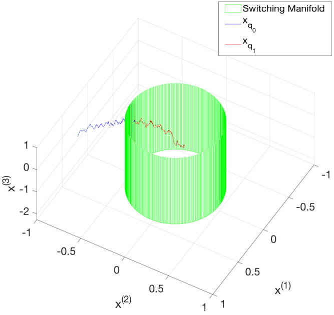

Optimal Input Structure

$$u_t^o = \text{Function}(t,\ x_t, \underset{\Im^t}{\mathbb{E}}^{\boldsymbol{u}}\{\tau\}, \underset{\Im^t}{\mathbb{E}}^{\boldsymbol{u}}\{x_{\tau-}\})$$

- For $t \geq \tau$ the information of $\underset{\Im^t}{\mathbb{E}}\{\tau\}$, $\underset{\Im^t}{\mathbb{E}}\{x_{\tau-}\}$ are contained in the hybrid state $h_t \equiv (q_1, x_t)$.
- For $t < \tau$ the values of $\underset{\Im^t}{\mathbb{E}}\{\tau\}$ and $\underset{\Im^t}{\mathbb{E}}\{x_{\tau-}\}$ are parts of predictions.

Presentation Outline

From Information (History) to State

- Hybrid State
- Hybrid Input

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The Missing Component

- Process–Process Duality: Adjoint Process (Minimum Principle)
- Measure–Function Duality: Value Function (Dynamic Programming)

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Computation and Implementation

- Linear Dynamics and Quadratic Costs
- Polynomial Dynamics and Costs
- Generally Nonlinear Dynamics and Costs

Part I

From History to the Notion of Hybrid State

Flow of a Dynamic System

$$(\text{State})_t = \text{Flow}\left(t; t_0, (\text{State})_{t_0}, [\text{Input}]_{t_0}^{t}\right)$$

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Deterministic System

$$h_t = \varphi\left(t; t_0, h_{t_0}, I_{t_0}^{t}\right) = \varphi\left(t; s, \varphi\left(s; t_0, h_{t_0}, I_{t_0}^{s}\right), I_s^t\right)$$

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Stochastic System

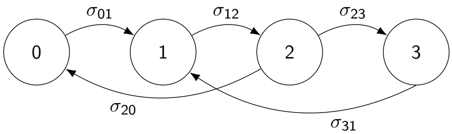
$$(\Omega, \Im, \Im^t, P) : \text{Filtered Probability Space}$$

$$P\Big(h_t \in B_h \;\Big|\; t_0, h_{t_0}, I_{t_0}^{t}\Big) = \int P\Big(h_t \in B_h \;\Big|\; s, h_s, I_s^{t}\Big) dP\Big(h_s \;\Big|\; t_0, h_{t_0}, I_{t_0}^{s}\Big)$$

The Notion of Hybrid State

$$h_t = \begin{bmatrix} q_t \\ x_t \end{bmatrix}, \qquad q_t \in Q, \;\; |Q| < \infty; \quad x_t \in \mathbb{R}^{n_q}$$

Discrete Dynamics - Finite Automata



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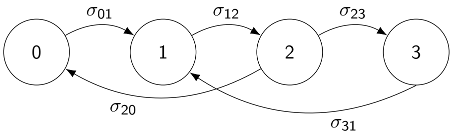

Continuous Dynamics

Fix $\boldsymbol{I}$, t, h_t. Then for infinitesimal increment(s): dt (and dw):

$$dh_t = d\begin{bmatrix} q_t \\ x_t \end{bmatrix} = \varphi\left(t; t_0, h_{t_0}, \boldsymbol{I}, \boldsymbol{w}\right) - \varphi\left(t+dt; t_0, h_{t_0}, \boldsymbol{I}, \boldsymbol{w}\right) = \begin{bmatrix} 0 \\ f_{q_t}(x_t, u_t)dt + g_{q_t}(x_t)dw \end{bmatrix}$$

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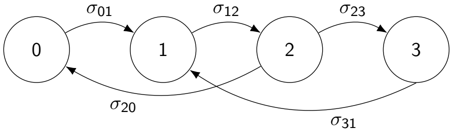

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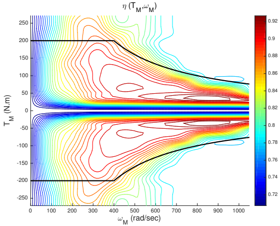
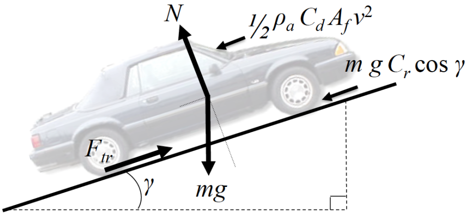
Hybrid Input

- Discrete input σ interacts with (activates) the discrete state q updates.
- Continuous input u interacts with the evolution of the continuous state x.

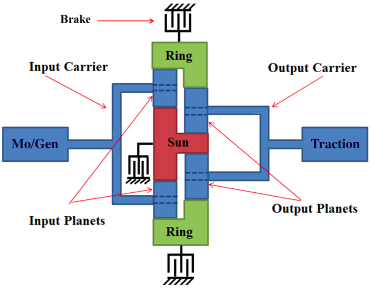
Electric Vehicle with Dual Planetary Transmission

Vehicle and Battery



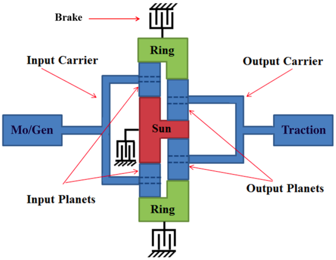


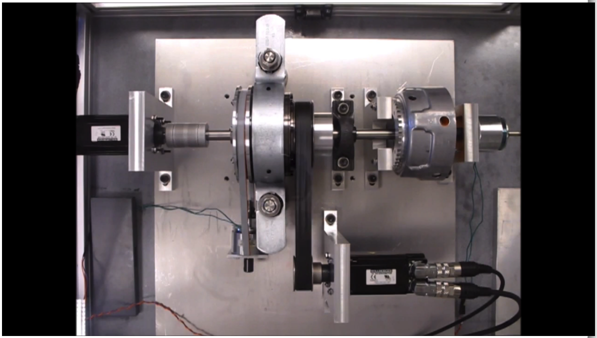
Transmission



Hybrid Input – Discrete Component

Dual Planetary Transmission [Patent US 9,702,438 B2]





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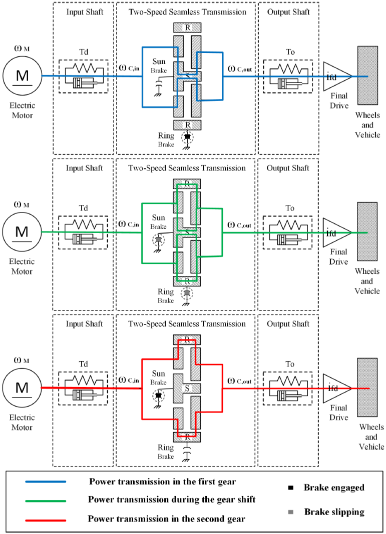

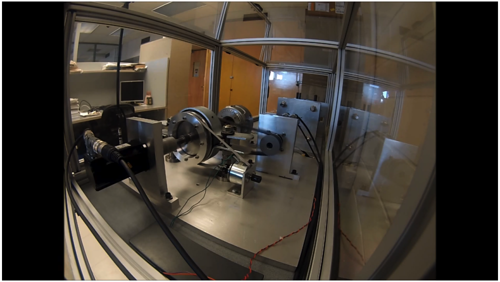

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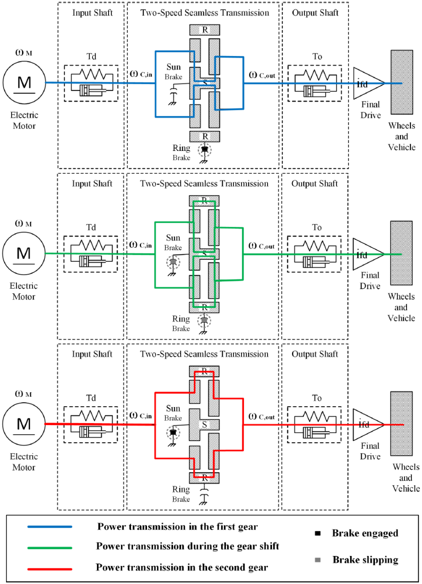

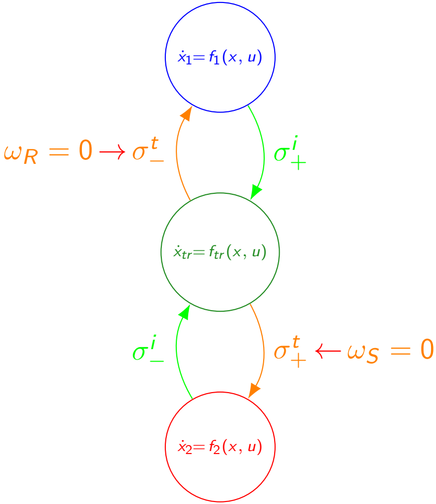

Controlled (Free) Switching

Autonomous (Uncontrolled) Switching

Stochastic Hybrid Systems

$$\left(\Omega, \Im, \Im^t, P\right)$$

Continuous Dynamics:

$$dx_{q_i}\left(t\right) = f_{q_i}\left(x_{q_i}\left(t\right), u_{q_i}\left(t\right)\right) dt + g_{q_i}\left(x_{q_i}\left(t\right)\right) dw, \quad t \in [t_i, t_{i+1})$$

Discrete Dynamics:

$$q\left(t_j\right) = \Gamma\left(q\left(t_j-\right), x_{q_{j-1}}(t_j-), \sigma_{q_{j-1}q_j}\right)$$

Switching Manifold and Jump Transition Map:

$$m_{q_{j-1}q_j}\left(x_{q_{j-1}}\left(t_j-\right)\right) \overset{a.s.}{=} 0,$$

$$x_{q_j}\left(t_j\right) = \xi_{\sigma_{q_{j-1}q_j}}\left(x_{q_{j-1}}\left(t_j-\right)\right) \equiv \xi_{\sigma_{q_{j-1}q_j}}\left(\lim_{t \uparrow t_j} x_{q_{j-1}}\left(t\right)\right)$$

Assumptions on Diffusions:

$$g_p\left(\xi_{\sigma_{q,p}}\left(x\right)\right) = \xi_{\sigma_{q,p}}\left(g_q\left(x\right)\right),$$

$$\left\langle g_q\left(x\right), \nabla m_{q,p}\left(x\right)\right\rangle = 0$$

Stochastic Hybrid Optimal Control Problem

Total Cost

$$J\left(t_0, t_f, h_0, L; I_L\right) := \mathbb{E}\left\{\sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}\left(s\right), u\left(s\right)\right) ds + \sum_{j=1}^{L} c_{\sigma_{q_{j-1}q_j}}\left(t_j, x_{q_{j-1}}\left(t_j-\right)\right) + h\left(x_{q_L}\left(t_f\right)\right)\right\}$$

Cost-to-Go

$$J\left(t, q, x, L-j+1; I_{L-j+1}\right) = \mathop{\mathbb{E}}_{\Im^t}\left\{\int_{t}^{t_j} l_q\left(x, u\right) ds + \sum_{i=j}^{L} c_{\sigma_{q_{i-1}q_i}}\left(t_i, x_{q_{i-1}}\left(t_i-\right)\right) + \sum_{i=j}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}\left(s\right), u\left(s\right)\right) ds + h\left(x_{q_L}\left(t_f\right)\right)\right\}$$

Value Function

$$V\left(t, q, x, L-j+1\right) = \inf_{I_{L-j+1}}\left\{\mathop{\mathbb{E}}_{\Im^t}\left\{J\left(t, q, x, L-j+1; I_{L-j+1}\right)\right\}\right\}$$

Part II

Duality Relationships

Process–Process Duality

Itô's Lemma

$$dZ(t) = b(t)\,dt + \sigma(t)\,dw(t)$$
$$d\hat{Z}(t) = \hat{b}(t)\,dt + \hat{\sigma}(t)\,dw(t)$$

Then for $\tau_2 \geq \tau_1$:

$$\begin{aligned}\left\langle Z(\tau_2), \hat{Z}(\tau_2)\right\rangle = \left\langle Z(\tau_1), \hat{Z}(\tau_1)\right\rangle \\ + \int_{\tau_1}^{\tau_2} \left\{\left\langle Z(s), \hat{b}(s)\right\rangle + \left\langle b(s), \hat{Z}(s)\right\rangle + \langle \sigma(s), \hat{\sigma}(s)\rangle\right\} ds \\ + \int_{\tau_1}^{\tau_2} \left\{\left\langle \sigma(s), \hat{Z}(s)\right\rangle + \langle Z(s), \hat{\sigma}(s)\rangle\right\} dw(s)\end{aligned}$$

Process–Process Duality

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+ \int_{\tau_1}^{\tau_2} \left\{\left\langle \sigma(s), \hat{Z}(s)\right\rangle + \left\langle Z(s), \hat{\sigma}(s)\right\rangle\right\} dw(s)
\end{aligned}
$$

Processes Z and $\hat{Z}$ are adjoint pairs if

$$
\underset{\Im^t}{\mathbb{E}} \left\langle Z(\tau_2), \hat{Z}(\tau_2)\right\rangle \overset{a.s.}{=} \underset{\Im^t}{\mathbb{E}} \left\langle Z(\tau_1), \hat{Z}(\tau_1)\right\rangle, \qquad t \in [t_0, \infty)
$$

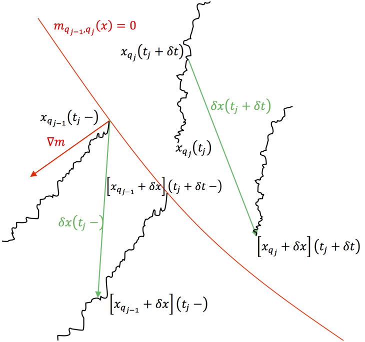
Process–Process Duality

Needle Variation

$$u^{\epsilon}(s) = \begin{cases} u^{o}_{q_0}(s) & \text{if} \quad t_0 \le s < t \\ v & \text{if} \quad t \le s < t+\epsilon \\ u^{o}_{q_0}(s) & \text{if} \quad t+\epsilon \le s < \tau - \delta^{\epsilon} \\ u^{o}_{q_1}(\tau) & \text{if} \quad \tau - \delta^{\epsilon} \le s < \tau \\ u^{o}_{q_1}(s) & \text{if} \quad \tau \le s \le t_f \end{cases} .$$

$$y(s) := \lim_{\epsilon \to 0} \frac{x^{\epsilon}_{q_i}(s) - x^{o}_{q_i}(s)}{\epsilon}$$

$$y(\tau) \overset{a.s.}{=} \nabla\xi y(\tau-) + \frac{\nabla m^T y(\tau-)}{\nabla m^T f_{q_0}} \left(f_{q_1} - \nabla\xi f_{q_0}\right)$$



Process–Process Duality

Duality of (z_s, y_s) and $(1, \lambda_s)$ in the Stochastic Minimum Principle

$$d\begin{bmatrix} z_s \\ y_s \end{bmatrix} = \begin{bmatrix} \frac{\partial l_{q_i}\left(x_s^o, u_s^o\right)}{\partial x_s} y_s \\ \frac{\partial f_{q_i}\left(x_s^o, u_s^o\right)}{\partial x_s} y_s \end{bmatrix} ds + \begin{bmatrix} 0 \\ \frac{\partial g_{q_i}\left(x_s^o\right)}{\partial x_s} y_s \end{bmatrix} dw$$

$$d\begin{bmatrix} 1 \\ \lambda_s \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\partial l_{q_i}\left(x_s^o, u_s^o\right)}{\partial x_s} - \left[\frac{\partial f_{q_i}\left(x_s^o, u_s^o\right)}{\partial x_s}\right]^T \lambda_s \end{bmatrix} ds + \begin{bmatrix} 0 \\ K_s \end{bmatrix} dw$$

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- (z_s, y_s) are forward linear processes resulted from variations around optimal processes
- $(1, \lambda_s)$ are backward linear processes, adjoint to the future values of (z_s, y_s) by

$$\underset{\Im^t}{\mathbb{E}}\left[z_{\tau_2} + \langle y_{\tau_2}, \lambda_{\tau_2}\rangle\right] = \underset{\Im^t}{\mathbb{E}}\left[z_{\tau_1} + \langle y_{\tau_1}, \lambda_{\tau_1}\rangle\right]$$

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$$\underset{\mathfrak{I}^t}{\mathbb{E}}\left[z_{\tau_2} + \langle y_{\tau_2}, \lambda_{\tau_2} \rangle\right] = \underset{\mathfrak{I}^t}{\mathbb{E}}\left[z_{\tau_1} + \langle y_{\tau_1}, \lambda_{\tau_1} \rangle\right]$$

In particular, the positivity of cost variations over the cone of variations (z_s, y_s) translates into those on Hamiltonian functions in $(1, \lambda_s)$

Stochastic Hybrid Minimum Principle (SHMP) [CDC 2016]

$$H_q\left(x_q, u_q, \lambda_q, K_q\right) := l_q\left(x_q, u_q\right) + \lambda_q^T f_q\left(x_q, u_q\right) + \mathrm{tr}\left[K_q^T g_q\left(x_q\right)\right]$$

Stochastic Hybrid Minimum Principle (SHMP) [CDC 2016]

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Hamiltonian Minimization

$$u_t^o = \underset{u_q(t)\in U_{q_t}}{\arg\inf}\ H_{q_t}\left(x_q^o(t), u_q(t), \lambda_q^o(t), K_q^o(t)\right)$$

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Hamiltonian Canonical Equations

$$dx_q{}^o = \frac{\partial H_{q^o}}{\partial \lambda_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dt + \frac{\partial H_{q^o}}{\partial K_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dw,$$

$$d\lambda_q{}^o = -\frac{\partial H_{q^o}}{\partial x_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dt + K_q^o dw,$$

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State Boundary Conditions

$$x_{q_0}^o\left(t_0\right) = x_0, \qquad x_{q_j}^o\left(t_j\right) \overset{a.s.}{=} \xi_{\sigma_{q_{j-1},q_j}}\left(x_{q_{j-1}}^o\left(t_j-\right)\right)$$

Stochastic Hybrid Minimum Principle (SHMP) [CDC 2016]

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$$u_t^o = \underset{u_q(t)\in U_{q_t}}{\arg\inf}\; H_{q_t}\left(x_q^o(t), u_q(t), \lambda_q^o(t), K_q^o(t)\right)$$

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State Boundary Conditions

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Adjoint State Boundary Conditions

$$\lambda_{q_L}^o\left(t_f\right) \overset{a.s.}{=} \frac{\partial h\left(x_{q_L}^o\left(t_f\right)\right)}{\partial x_{q_L}}, \qquad \lambda_{q_{j-1}}^o\left(t_j\right) \overset{a.s.}{=} \left[\frac{\partial \xi_{\sigma_{q_{j-1},q_j}}}{\partial x_{q_{j-1}}}\right]^T \lambda_{q_j}^o\left(t_j+\right) + p\,\frac{\partial m_{q_{j-1},q_j}}{\partial x_{q_{j-1}}} + \frac{\partial c_{\sigma_{q_{j-1},q_j}}}{\partial x_{q_{j-1}}}$$

Stochastic Hybrid Minimum Principle (SHMP) [CDC 2016]

$$H_q\left(x_q, u_q, \lambda_q, K_q\right) := l_q\left(x_q, u_q\right) + \lambda_q^T f_q\left(x_q, u_q\right) + \mathrm{tr}\left[K_q^T g_q\left(x_q\right)\right]$$

Hamiltonian Minimization

$$u_t^o = \arg\inf_{u_q(t)\in U_{q_t}} H_{q_t}\left(x_q^o(t), u_q(t), \lambda_q^o(t), K_q^o(t)\right)$$

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$$dx_q{}^o = \frac{\partial H_{q^o}}{\partial \lambda_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dt + \frac{\partial H_{q^o}}{\partial K_q}\left(x_q^o, u_q^o, \lambda_q^o, K_q^o\right) dw,$$

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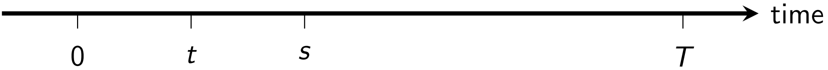
Hamiltonian Boundary Conditions

$$H_{q_{j-1}} - \mathrm{tr}\left[K_{q_{j-1}}^o{}^T g_{q_{j-1}}\right]\Big|_{t_j-} = H_{q_j} - \mathrm{tr}\left[K_{q_j}^o{}^T g_{q_j}\right]\Big|_{t_j+}$$

Killed Markov Processes [CDC 2019 (accepted)]



Dynamics (Itô differential equation)

$$dx_s = f\left(x_s, u_s\right) ds + g(x_s) dw_s, \qquad x \in X^0$$

$$dx_s = 0 \; ds + 0 \; dw_s, \qquad x \in X^{\partial}$$

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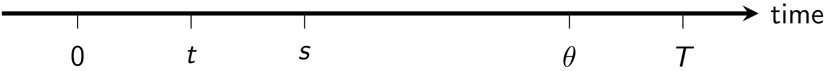

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First Arrival Time θ on the Boundary X^∂

$$x_{\theta-} \equiv x_\theta \in X^\partial$$

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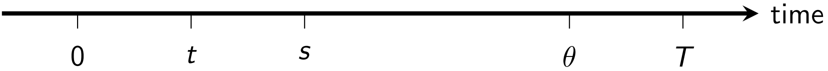

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First Arrival Time θ on the Boundary X^∂

$$x_{\theta-} \equiv x_\theta \in X^\partial$$

Total Cost

$$J\left(t, x, \boldsymbol{u}\right) = \mathbb{E}^{\boldsymbol{u}}_{t,x} \left\{ \int_t^{\min\{\theta, T\}} l\left(x_s, u_s\right) ds + \mathbb{I}_{[t,T)}(\theta) \cdot \ell\left(\theta, x_\theta\right) + \mathbb{I}^c_{[t,T)}(\theta) \cdot L\left(x_T\right) \right\},$$

Occupation Measures

Input-State-Time Occupation Measure

$$\mu^{\boldsymbol{u}}\left(B_t, B_x^0, B_u\right) := \mathbb{E}_{t,x}^{\boldsymbol{u}} \int_{B_t \cap [t,T)} \mathbb{I}_{B_x^0}(x_s) \cdot \mathbb{I}_{B_u}(u_s)\, ds,$$

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Switching State-Time Occupation Measure

$$\eta^{\boldsymbol{u}}\left(B_t, B_x^{\partial}\right) := P^{\boldsymbol{u}}_{t,x}\Big(\mathbb{I}_{[t,T)\cap B_t}(\theta) = 1\ ,\ x^{\boldsymbol{u}}_{\theta-} \in B_x^{\partial}\Big),$$

Occupation Measures

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Terminal State Occupation Measure

$$\kappa^{\boldsymbol{u}}\left(B_x\right) := P^{\boldsymbol{u}}_{t,x}\left(\mathbb{I}_{[t,T)}(\theta) = 0\ ,\ x^{\boldsymbol{u}}_{T} \in B_x\right).$$

Occupation Measures

Input-State-Time Occupation Measure

$$\mu^{\boldsymbol{u}}\left(B_t, B_x^0, B_u\right) := \mathbb{E}_{t,x}^{\boldsymbol{u}} \int_{B_t \cap [t,T)} \mathbb{I}_{B_x^0}(x_s) \cdot \mathbb{I}_{B_u}(u_s)\, ds,$$

Switching State-Time Occupation Measure

$$\eta^{\boldsymbol{u}}\left(B_t, B_x^{\partial}\right) := P_{t,x}^{\boldsymbol{u}}\left(\mathbb{I}_{[t,T)\cap B_t}(\theta) = 1\ ,\ x_{\theta-}^{\boldsymbol{u}} \in B_x^{\partial}\right),$$

Terminal State Occupation Measure

$$\kappa^{\boldsymbol{u}}\left(B_x\right) := P_{t,x}^{\boldsymbol{u}}\left(\mathbb{I}_{[t,T)}(\theta) = 0\ ,\ x_T^{\boldsymbol{u}} \in B_x\right).$$

Defining $\mathcal{M}_S := \{(\mu^{\boldsymbol{u}}, \eta^{\boldsymbol{u}}, \kappa^{\boldsymbol{u}}) : \boldsymbol{u} \in \mathcal{U}\}$ we obtain:

$$V(t,x) = \inf_{(\mu^{\boldsymbol{u}}, \eta^{\boldsymbol{u}}, \kappa^{\boldsymbol{u}}) \in \mathcal{M}_S} \left\{\left\langle l, \mu^{\boldsymbol{u}}\right\rangle + \left\langle \ell, \eta^{\boldsymbol{u}}\right\rangle + \left\langle L, \kappa^{\boldsymbol{u}}\right\rangle\right\}$$

Reformulation

Infinitesimal Operator

$$\mathcal{A}^y v(t,x) = \frac{\partial v(t,x)}{\partial t} + \left\langle f(x,y), \frac{\partial v(t,x)}{\partial x} \right\rangle + \frac{1}{2}\,\mathrm{tr}\left(g^T g\, \frac{\partial^2 v(t,x)}{\partial x^2} \right)$$

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Dynkin Formula

$$\begin{aligned} \mathbb{E}^{\boldsymbol{u}}_{t,x} v(\tau, x_\tau) &= \mathbb{E}^{\boldsymbol{u}}_{t,x} \Big\{ \mathbb{I}_{[t,T)}(\theta) \cdot v(\theta, x_\theta) + \mathbb{I}^{c}_{[t,T)}(\theta) \cdot v(T, x_T) \Big\} \\ &= v(t,x) + \mathbb{E}^{\boldsymbol{u}}_{t,x} \int_{t}^{\min\{\theta, T\}} \mathcal{A}^{u_s} v(s, x_s)\, ds \end{aligned}$$

Reformulation

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Defining the adjoint $\mathcal{A}^*$ to be one satisfying $\langle \mathcal{A} v, \mu \rangle = \langle v, \mathcal{A}^* \mu \rangle$ we obtain

$$\eta^{\boldsymbol{u}} + \kappa^{\boldsymbol{u}} = \delta_{t,x} + \mathcal{A}^* \mu^{\boldsymbol{u}}$$

Weak Problem

A Convex Subset of Signed Measures

Define $\mathcal{M}_W := \mathcal{M}_{PB} \cap \mathcal{M}_{\mathcal{A}}$, with

$$\mathcal{M}_{PB} := \Big\{ M \equiv (\mu, \eta, \kappa) \in \mathfrak{M}_+ \left([0, T] \times X \times U\right) : \|M\| \leq T - t + 1 \Big\}$$
$$\mathcal{M}_{\mathcal{A}} := \Big\{ M \equiv (\mu, \eta, \kappa) \in \mathfrak{M}_\pm \left([0, T] \times X \times U\right) : \eta + \kappa = \delta_{t,x} + \mathcal{A}^* \mu \Big\}$$

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Weak Problem

$$W(t, x) := \min_{M \in \mathcal{M}_W} \langle l, \mu \rangle + \langle \ell, \eta \rangle + \langle L, \kappa \rangle \leq V(t, x)$$

Weak Problem

A Convex Subset of Signed Measures

Define $\mathcal{M}_W := \mathcal{M}_{PB} \cap \mathcal{M}_{\mathcal{A}}$, with

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Weak Problem

$$W(t, x) := \min_{M \in \mathcal{M}_W} \langle l, \mu \rangle + \langle \ell, \eta \rangle + \langle L, \kappa \rangle \leq V(t, x)$$

Dual Problem

$$W(t, x) = \sup \Big\{ v(t, x) : v \in C^2 ([0, T] \times X),$$
$$\mathcal{A}v + l \geq 0, \quad v^\partial - \ell \leq 0, \quad v^T - L \leq 0 \Big\}$$

Equivalence of the Strong and Weak Problems

$$W(t,x) = \min_{M \in \mathfrak{M}_{\pm}([0,T]\times X \times U)} h_1(M) - h_2(M)$$

$$h_1(M) := \begin{cases} \langle l, \mu \rangle + \langle \ell, \eta \rangle + \langle L, \kappa \rangle & \text{if } M \equiv (\mu, \eta, \kappa) \in \mathcal{M}_{PB} \\ +\infty & \text{otherwise} \end{cases}$$

$$h_2(M) := \begin{cases} 0 & \text{if } M \equiv (\mu, \eta, \kappa) \in \mathcal{M}_{\mathcal{A}} \\ -\infty & \text{otherwise} \end{cases}$$

Legendre-Fenchel Transform

$$h_1^*(c) := \sup_{M \in \mathcal{M}_{PB}} \left\{ \langle c^0, \mu \rangle + \langle c^\partial, \eta \rangle + \langle c^T, \kappa \rangle - \langle l, \mu \rangle + \langle \ell, \eta \rangle + \langle L, \kappa \rangle \right\}$$

$$h_2^*(c) := \inf_{M \in \mathcal{M}_{\mathcal{A}}} \left\{ \langle c^0, \mu \rangle + \langle c^\partial, \eta \rangle + \langle c^T, \kappa \rangle \right\} = \begin{cases} \lim_{i \to \infty} v_i(t,x) & \text{if } \begin{cases} c^0 = -\lim_{i\to\infty} \mathcal{A} v_i \\ c^\partial = \lim_{i\to\infty} v_i^\partial \\ c^T = \lim_{i\to\infty} v_i^T \end{cases} \\ -\infty & \text{otherwise} \end{cases}$$

Part III

Numerical Algorithms

Polynomial Approximation

Stone–Weierstrass Theorem

Over the compact domain $[0, T] \times X \subset \mathbb{R}^{n+1}$, the algebra of polynomials, $\mathbb{R}\left[s, x\right]$, is dense in $C\left([0, T] \times X\right)$ and, consequently, in $C^2\left([0, T] \times X\right)$.

Polynomial Approximation

Stone–Weierstrass Theorem

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Polynomial Approximation Theorem

$$V(t_0, x_0) = \sup\Big\{v\left(t_0, x_0\right) : v \in C^2\left([0, T] \times X\right), \\ \mathcal{A}v + l \geq 0, \quad v^{\partial} - \ell \leq 0, \quad v^{T} - L \leq 0\Big\}$$

Polynomial Approximation

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Putinar's Positivstellensatz

If $w(x) \in \mathbb{R}[x]$ is strictly positive on X where

$$X := \{x \in \mathbb{R}^n : h_X^{(i)}(x) \geq 0, i = 1, \cdots, m\}$$

$$\Rightarrow w(x) = w^{(0)}(x) + \sum_{i=1}^{m} w^{(i)}(x) \cdot h_X^{(i)}(x)$$

Polynomial Approximation

Stone–Weierstrass Theorem

Over the compact domain $[0, T] \times X \subset \mathbb{R}^{n+1}$, the algebra of polynomials, $\mathbb{R}\big[s, x\big]$, is dense in $C\left([0, T] \times X\right)$ and, consequently, in $C^2\left([0, T] \times X\right)$.

Polynomial Approximation Theorem

$$V(t_0, x_0) = \sup\Big\{v\left(t_0, x_0\right) : v \in \mathbb{R}[t, x],$$
$$\mathcal{A}v + l \in Q_{2k}(h_T, h_X, h_U), \quad \ell - v^{\partial} \in Q_{2k}(h_T, h_X), \quad L - v^{T} \in Q_{2k}(h_X)\Big\}$$

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$$\Rightarrow w(x) = w^{(0)}(x) + \sum_{i=1}^{m} w^{(i)}(x) \cdot h_X^{(i)}(x)$$

LQG Hybrid Optimal Control Problems

Hybrid Dynamics:

$$dx_{q_i} = \left(A_{q_i} x_{q_i} + B_{q_i} u_{q_i}\right) dt + G_{q_i} dw, \qquad t \in \left[t_i, t_{i+1}\right),$$

Hybrid Cost:

$$J = \frac{1}{2}\mathbb{E}\left\{\sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} \left\|x_{q_i}(t)\right\|^2_{L_{q_i}} + \left\|u_{q_i}(t)\right\|^2_{R_{q_i}} dt + \left\|x_{q_L}(t_f)\right\|^2_{H_{q_L}}\right\}$$

Jump Transition Map:

$$x_{q_j}(t_j) = \Psi_{\sigma_j} x_{q_{j-1}}(t_j-) \equiv \Psi_{q_{j-1}q_j} x_{q_{j-1}}(t_j-)$$

Switching Manifolds:

$$m_{q_{i-1}q_i}\left(x_{q_{i-1}}(t_i-)\right) \equiv \frac{1}{2}\left(\left\|x_{q_{i-1}}(t_i-)\right\|^2_{M_{q_{i-1}q_i}} - r^2_{q_{i-1}q_i}\right) = 0,$$

Assumptions on Diffusions and Switching Manifolds:

$$G_{q_k} = \Psi_{q_{k-1}q_k} G_{q_{k-1}}, \qquad M_{q_{i-1}q_i} G_{q_i} = 0$$

Hybrid Optimal Control Solutions [In Preparation]

Optimal Feedback Input:

$$u^{o}_{q_i}(t) = -R^{-1}_{q_i} B^{T}_{q_i} \left(\Pi_{q_i}\left(t; \underset{\Im^t}{\mathbb{E}}(t_{i+1})\right) x_{q_i}(t) + s_{q_i}\left(t; \underset{\Im^t}{\mathbb{E}}\left(t_{i+1}, x_{q_i}(t_{i+1}-)\right)\right)\right)$$

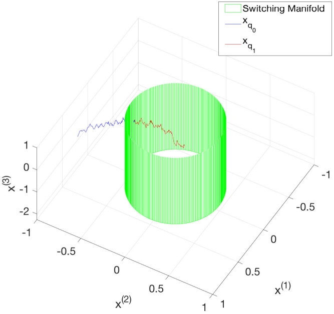
Hybrid Stochastic Riccati Equations

$$\begin{aligned}
&\dot{\Pi}_{q_i} = \Pi_{q_i} B_{q_i} R^{-1}_{q_i} B_{q_i} \Pi_{q_i} - \Pi_{q_i} A_{q_i} - A^{T}_{q_i} \Pi_{q_i} - L_{q_i},\\
&\dot{s}_{q_i} = -\left(A^{T}_{q_i} - \Pi_{q_i} B_{q_i} R^{-1}_{q_i} B^{T}_{q_i}\right) s_{q_i},\\
&\Pi_{q_L}(t_f) = H_{q_L},\\
&s_{q_L}(t_f; t_f, x_f) = 0\\
&\Pi_{q_{j-1}}(t_j; t_j) = \Psi^{T}_{\sigma_j} \Pi_{q_j}\left(t_j; \underset{\Im^{t_j}}{\mathbb{E}}(t_{j+1})\right) \Psi_{\sigma_j}\\
&s_{q_{j-1}}(t_j; t_j, x_{q_{j-1}}) = \Psi^{T}_{\sigma_j} s_{q_j}\left(t_j; \mathbb{E}\left(t_{j+1}, x_{q_j}(t_{j+1}-)\right)\right) + p_{(t_j, x_{q_{j-1}})} M_{\sigma_j} x_{q_{j-1}},\\
&\alpha_{(t_j, x_{q_{j-1}})} p^2 + \beta_{(t_j, x_{q_{j-1}})} p + \gamma_{(t_j, x_{q_{j-1}})} = 0.
\end{aligned}$$

Example with Switching Manifold

$$dx = \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \right) dt + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dw$$

$$dx = \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \right) dt + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dw$$



$$J = \frac{1}{2} \mathbb{E} \left\{ \int_{t_0}^{t_1} \left(u_{q_0}(t)\right)^2 dt + \int_{t_1}^{t_f} \left(u_{q_1}(t)\right)^2 dt + x_{q_1}(t_f)^T \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} x_{q_1}(t_f) \right\}$$

Hybrid LQG Example [Pakniyat, Caines, IFAC 2017]

Hybrid Dynamics:

$$q_1: \qquad dx_1 = \left(\frac{31}{16}x_1 + u_1\right) dt + g_1 dw,$$

$$q_2: \qquad dx_2 = \left(\frac{3}{8}x_2 + u_2\right) dt + g_2\, dw,$$

with $\quad g_1 = 1, \quad g_2 = \sqrt{2}g_1 = \sqrt{2}$

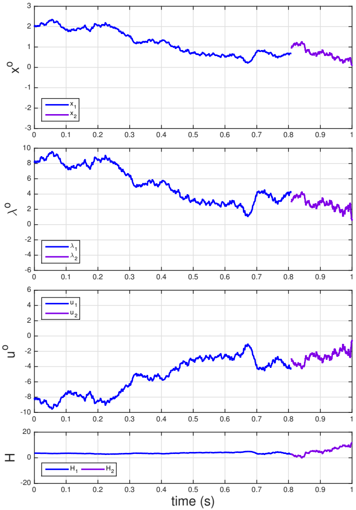
Controlled Switching Jump Transition Map

$$x_2\left(t_s\right) = \sqrt{2}\, x_1\left(t_s-\right)$$

Hybrid Cost:

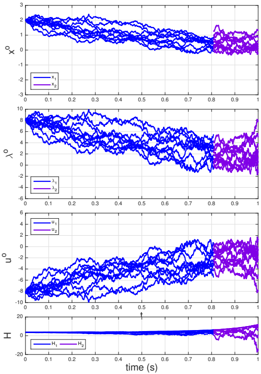
$$J\left(t_0, t_f, h_0, L; l_L\right)$$

$$= \mathbb{E}\left\{ \frac{1}{2}\int_{t_0}^{t_s} \left(\left(u_1\left(t\right)\right)^2 + \frac{1}{2}\left(x_1\left(t\right)\right)^2\right) dt + \frac{1}{2}\int_{t_s}^{t_f} \left(\left(u_2\left(t\right)\right)^2 + \frac{1}{4}\left(x_2\left(t\right)\right)^2\right) dt + \frac{1}{2} \times 6\left(x_2\left(t_f\right)\right)^2 \right\}$$

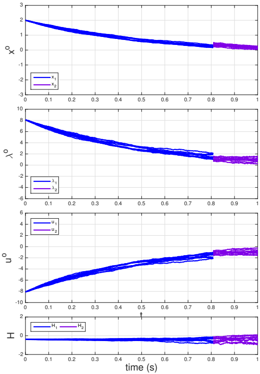


Hybrid LQG Example [Pakniyat & Caines, IFAC 2017]

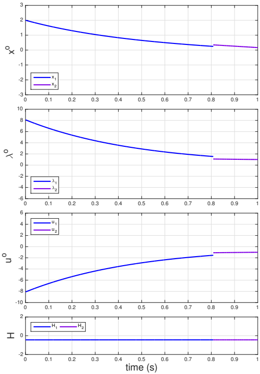
Ten sample paths for continuous states, adjoint processes, continuous inputs and Hamiltonians in the example with $t_f = 1$ and $x_0 = 2$, and



$g_1 = 1,\ g_2 = \sqrt{2}$



$g_1 = 0.1,\ g_2 = 0.1\sqrt{2}$



$g_1 = g_2 = 0$

Generally Nonlinear: PDE–BSDE Duality (Feynman-Kac)

Partial Differential Equation (PDE)

$$V_t + \frac{1}{2} tr\Big(g(t,x)g(t,x)^T V_{xx}\Big) + V_x^T f(t,x) + h(t,x,V,g(t,x)^T V_x) = 0, \qquad (t,x) \in [0,T] \times X$$

$$V(T,x) = L(x), \qquad x \in X$$

$$V(\tau,z) = \ell(\tau,z), \qquad (\tau,z) \in [0,T] \times X^{\partial}$$

Backward Stochastic Differential Equation (BSDE)

$$dY_s = -h\left(s, X_s, Y_s, Z_s\right) ds + Z_s dw_s$$

$$Y_\tau = \begin{cases} \ell\left(\tau, X_\tau\right), & X_\tau \in X^{\partial} \\ L\left(X_T\right), & \tau = T, X_\tau \in X^0 \end{cases}$$

Feynman-Kac Representation

$$Y_t = \mathbb{E}_{t,x} V(t,x)$$

$$Z_t = \mathbb{E}_{t,x} g(t,x)^T V_x(t,x)$$

Summary

From Information (History) to State

- Hybrid State
- Hybrid Input

From Prediction to the Missing Component

- The [Stochastic] Hybrid Minimum Principle: Adjoint Process
- [Stochastic] Hybrid Dynamic Programming: Value Function

Derivation

- Process–Process Duality
- Measure–Function Duality

Computation and Implementation

- Generally Nonlinear Dynamics and Costs
- Linear Dynamics and Quadratic Costs
- Polynomial Dynamics and Costs

Relevant Publications

[TAC2017] A. Pakniyat and P. E. Caines, "On the Relation between the Minimum Principle and Dynamic Programming for Classical and Hybrid Control Systems," IEEE Transactions on Automatic Control, vol. 62, no. 9, pp. 4347–4362, 2017

[NAHS2017] A. Pakniyat and P. E. Caines, "Hybrid Optimal Control of an Electric Vehicle with a Dual-Planetary Transmission," Nonlinear Analysis: Hybrid Systems, vol. 25, pp. 263–282, 2017

[MMT2015] M. S. R. Mousavi, A. Pakniyat, T. Wang, and B. Boulet, "Seamless Dual Brake Transmission For Electric Vehicles: Design, Control and Experiment," Mechanism and Machine Theory, vol. 94, pp. 96–118, 2015

[Patent US 9,702,438 B2] B. Boulet, M. S. R. Mousavi, H. V. Alizadeh, and A. Pakniyat, "Seamless Transmission Systems and Methods for Electric Vehicles," Jul. 11 2017, US Patent US 9,702,438 B2

[CDC2019 (accepted)] A. Pakniyat and R. Vasudevan, "A Convex Duality Approach to Optimal Control of Killed Markov Processes," in Manuscript 2039 to appear in the Proceedings of the 58th IEEE Conference on Decision and Control, 2019

[CDC2017] D. Firoozi, A. Pakniyat, and P. E. Caines, "A Mean Field Game - Hybrid Systems Approach to Optimal Execution Problems in Finance with Stopping Times," in Proceedings of the 56th IEEE Conference on Decision and Control, Melbourne, Australia, 2017, pp. 433–441

[CDC2016] A. Pakniyat and P. E. Caines, "On the Stochastic Minimum Principle for Hybrid Systems," 2016, pp. 1139–1144

[IFAC2017] A. Pakniyat and P. E. Caines, "A Class of Linear Quadratic Gaussian Hybrid Optimal Control Problems with Realization–Independent Riccati Equations," 2017, pp. 2241–2246